# MOMENTS OF PRODUCTS OF $L$-FUNCTIONS

DISSERTATION

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ABSTRACT

We first consider questions on the distribution of the primes. Using the recent advancement towards the Prime $k$-tuple Conjecture by Maynard and Tao, we show how to produce infinitely many strings of consecutive primes satisfying specified congruence conditions. We answer an old question of Erdős and Turán by producing strings of consecutive primes whose successive gaps form an increasing (respectively decreasing) sequence. We also show that such strings exist whose successive gaps follow a certain divisibility pattern. Finally, for any coprime integers $a$ and $D \geq 1$, we refine a theorem of D. Shiu and find strings of consecutive primes of arbitrary length in the congruence class $a \bmod D$. These results were proved jointly with William D. Banks and Tristan Freiberg.

We next consider the vertical distribution of the nontrivial zeros of certain Dedekind zeta-functions. In particular, let $K$ be a quadratic number field, and let $\zeta_K(s)$ denote the Dedekind zeta-function attached to $K$. Using the mixed second moments of derivatives of $\zeta_K(s)$ on the critical line, we prove the existence of gaps between consecutive zeros of $\zeta_K(s)$ on the critical line which are at least $\sqrt{6} = 2.44949\ldots$ times the average spacing.

Finally, assuming the Generalized Riemann Hypothesis and some standard conjectures, we prove upper bounds for moments of arbitrary products of automorphic $L$-functions and for Dedekind zeta-functions of Galois number fields on the critical line. As an application, we use these bounds to estimate the variance of the coefficients of these zeta- and $L$-functions in short intervals. We also prove upper bounds for moments of products of central values of automorphic $L$-functions twisted by quadratic Dirichlet characters and averaged over fundamental discriminants. These results were proved jointly with Micah B. Milinovich.

DEDICATION

To Ian, for his adoration, friendship, and support.

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# 1 INTRODUCTION

*Prime numbers are the most basic objects in mathematics. They also are among the most mysterious, for after centuries of study, the structure of the set of prime numbers is still not well understood. Describing the distribution of primes is at the heart of much mathematics...*

– Andrew Granville [36]

## 1.1 Primes in Tuples

Let $p$ denote a prime. *Twin primes* are pairs of primes of the form $(p, p+2)$; examples include $(3, 5)$, $(11, 13)$, and $(41, 43)$. The Twin Prime Conjectures states that there are infinitely many such pairs of primes, but this is an open question. To date, the largest known twin primes are

$$(3756801695685 \cdot 2^{666,669} - 1, \quad 3756801695685 \cdot 2^{666,669} + 1).$$

In 1849, de Polignac [26] made the more general conjecture that for every natural number $n$, there are infinitely many pairs of primes of the form $(p, p+2n)$. Notice that the Twin Prime Conjecture is a special case of de Polignac's Conjecture. In April 2013, Zhang [97] proved that there are infinitely pairs of primes that are at most 70 million apart. During the summer of 2013, the constant 70 million was reduced via the online, collaborative polymath8 project (see [74, 75]) using ideas of Zhang and, subsequently, of Maynard [62] and

Tao[1]. As of June 20, 2014, the best unconditional result attained is that there are infinitely many pairs of primes at most 246 apart.

There are many other interesting questions concerning the distribution of the primes. For example, in 1948 Erdős and Turán [29] asked the following question.

**Question.** *Let $\{p_n\}$ denote the sequence of primes and $k$ be a natural number. Can the inequalities*

$$p_{n+1} - p_n < p_{n+2} - p_{n+1} < \cdots < p_{n+k} - p_{n+k-1}$$

*have infinitely many solutions for every fixed $k$?*

As a consequence of a recent result of Maynard and Tao (described below), William D. Banks, Tristan Freiberg, and I have answered this question in the affirmative.

In general, one may consider a $k$-tuple of linear forms in $\mathbb{Z}[x]$ and inquire as to whether or not the tuple can represent primes infinitely often. For example, consider the 3-tuple

$$\{x,\, x+2,\, x+4\}.$$

The choice of $x = 3$ produces the prime triple $\{3, 5, 7\}$. The 3-tuple $\{x,\, x+2,\, x+4\}$ cannot, however, represent primes for infinitely many integer values of $x$ since for every $x \in \mathbb{N}$, one of the entries in the tuple is always divisible by 3. In order to avoid such an impediment when searching for $k$-tuples which represent primes infinitely often, we introduce the following notion.

**Definition 1.1.1.** A $k$-tuple of linear forms in $\mathbb{Z}[x]$, denoted by

$$\mathcal{H}(x) := \{g_j x + h_j\}_{j=1}^{k},$$

[1] In [62], Maynard writes "Terence Tao (private communication) has independently proven Theorem 1.1 (with a slightly weaker bound) at much the same time."

is said to be *admissible* if the associated polynomial $f_{\mathcal{H}}(x) := \prod_{1\le j\le k}(g_jx+h_j)$ has no fixed prime divisor, that is, if the inequality

$$\#\{n \bmod p : f_{\mathcal{H}}(n) \equiv 0 \bmod p\} < p$$

holds for every prime number $p$.

The results given in Section 1.1.1 of this introduction require the entries of the admissible $k$-tuples to be distinct and positive for large values of $k$. To this end, we consider $k$-tuples for which

$$g_1,\dots,g_k > 0 \qquad \text{and} \qquad \prod_{1\le i<j\le k}(g_ih_j - g_jh_i) \ne 0. \tag{1.1.1}$$

One form of the Prime $k$-tuple Conjecture asserts that if $\mathcal{H}(x)$ is admissible and satisfies (1.1.1), then

$$\mathcal{H}(n) = \{g_jn + h_j\}_{j=1}^{k}$$

is a $k$-tuple of primes for infinitely many $n \in \mathbb{N}$. In November of 2013, Maynard [62] and Tao came very close to proving this form of the Prime $k$-tuple Conjecture. The following formulation of their remarkable theorem has been given by Granville [37, Theorem 6.2].

**Maynard–Tao Theorem.** *Let $\{g_jx+h_j\}_{j=1}^{k}$ be an admissible $k$-tuple satisfying* (1.1.1)*. For any natural number $m \ge 2$, there is a number $k_m$, depending only on $m$, such that for every integer $k \ge k_m$, the $k$-tuple $\{g_jn+h_j\}_{j=1}^{k}$ contains $m$ primes for infinitely many $n \in \mathbb{N}$. Moreover, one can take $k_m$ to be any number such that $k_m \log k_m > \mathrm{e}^{8m+4}$.*

For a thorough overview of these problems on gaps between primes and the ideas of Zhang and Maynard, we refer the reader to Granville's survey article [37].

## 1.1 Consecutive Primes in Tuples

The following theorem establishes the existence of $m$-tuples that infinitely often represent strings of consecutive prime numbers. This theorem and the three succeeding corollaries were proved in collaboration with William D. Banks and Tristan Freiberg. (See [4].)

**Theorem 1.1.2.** *Let $m, k \in \mathbb{N}$ with $m \geq 2$ and $k \geq k_m$, where $k_m \log k_m > \mathrm{e}^{8m+4}$, as in the Maynard–Tao Theorem. Let $b_1, \dots, b_k$ be distinct integers such that $\{x + b_j\}_{j=1}^k$ is admissible, and let $g$ be any positive integer coprime with $b_1 \cdots b_k$. Then, for some subset $\{h_1, \dots, h_m\} \subseteq \{b_1, \dots, b_k\}$, there are infinitely many $n \in \mathbb{N}$ such that $gn + h_1, \dots, gn + h_m$ are consecutive primes.*

Theorem 1.1.2 has various applications to the study of gaps between consecutive primes. In order to state our results more easily, let us call a sequence $(\delta_j)_{j=1}^m$ of positive integers a *run of consecutive prime gaps* if

$$\delta_j := d_{r+j} := p_{r+j+1} - p_{r+j} \qquad (1 \leq j \leq m)$$

for some natural number $r$, where $p_n$ denotes the $n$th prime. The following corollary of Theorem 1.1.2 answers an old question of Erdős and Turán [29] (see also Erdős [28] and Guy [38, A11]).

**Corollary 1.1.3.** *For every natural number $m \geq 2$, there are infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps with $\delta_1 < \cdots < \delta_m$ and infinitely many runs with $\delta_1 > \cdots > \delta_m$.*

In Chapter 2, we prove Corollary 1.1.3 by constructing infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps with

$$\delta_1 + \cdots + \delta_{j-1} < \delta_j \qquad (2 \leq j \leq m),$$

and infinitely many runs with

$$\delta_j > \delta_{j+1} + \cdots + \delta_m \qquad (1 \le j \le m-1).$$

Using a similar argument, we can also impose a divisibility requirement among gaps between consecutive primes.

**Corollary 1.1.4.** *For every natural number $m \ge 2$ there exist infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps such that $\delta_j \mid \delta_{j+1}$ for $1 \le j \le m-1$, and infinitely many runs for which $\delta_{j+1} \mid \delta_j$ for $1 \le j \le m-1$.*

As in the previous corollary, we can actually prove a bit more. Indeed, in the proof of Corollary 1.1.4 given in Chapter 2, we construct infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps with

$$\delta_1 \cdots \delta_{j-1} \mid \delta_j$$

for $2 \le j \le m$ and infinitely many runs with

$$\delta_m \delta_{m-1} \cdots \delta_{j+1} \mid \delta_j$$

for $1 \le j \le m-1$.

In 1920, Chowla conjectured that for $D \ge 3$ and $(a, D) = 1$, there are infinitely many pairs of consecutive primes $p_r$ and $p_{r+1}$ with

$$p_r \equiv p_{r+1} \equiv a \bmod D.$$

(See also [38, A4].) In 1997, D. Shiu [83] proved this conjecture for all $a$ and $D$ with $(a, D) = 1$. Moreover, he proved the following theorem on consecutive primes in a given congruence class.

**Theorem** (D. Shiu)**.** *Let a and D be coprime integers. For every natural number $m \geq 2$, there are infinitely many $r \in \mathbb{N}$ such that*

$$p_{r+1} \equiv p_{r+2} \equiv \cdots \equiv p_{r+m} \equiv a \bmod D.$$

Our final application of Theorem 1.1.2 is the following extension of Shiu's theorem.

**Corollary 1.1.5.** *Let a and D be coprime integers. For every natural number $m \geq 2$, there are infinitely many $r \in \mathbb{N}$ such that*

$$p_{r+1} \equiv p_{r+2} \equiv \cdots \equiv p_{r+m} \equiv a \bmod D$$

*and $p_{r+m} - p_{r+1} \leq C_m$, where $C_m$ is a constant depending on $m$ and $D$.*

For infinitely many $n \in \mathbb{N}$, let $\{Dn + h_1, \ldots, Dn + h_m\}$ be the string of consecutive primes guaranteed by Theorem 1.1.2. Then in our proof of Corollary 1.1.5, we show that $C_m = h_m - h_1$.

## 1.2 The Prime Number Theorem and the Riemann Zeta-function

*The Riemann zeta-function is a riddle par excellence. It is natural to fall in love with such a riddle, and then get disappointed by seeing that not much progress is to be hoped for quickly.*

– Cem Yıldırım [94]

In 1737, Euler proved that the sum of the reciprocals of the primes diverges. A key component in this proof was his discovery that, for all real $s > 1$,

$$\sum_{n=0}^{\infty} \frac{1}{n^s} = \prod_{\substack{p \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1.2.1}$$

The above sum is called a *Dirichlet series*, and the product is called an *Euler product.* Note that the infinitude of primes is a consequence of this equality. To study the distribution of the primes, we consider the prime counting function

$$\pi(x) := \sum_{p \le x} 1.$$

Gauss [31] and Legendre [57] independently conjectured the asymptotic behavior of $\pi(x)$ as $x$ grows arbitrarily large. Their conjectures imply that the ratio

$$\frac{\pi(x)}{x/\log x}$$

approaches 1 as $x$ tends to infinity. In 1896, Hadamard [39] and de la Vallée Poussin [90] independently proved this conjecture, known as the Prime Number Theorem. The analytic proof of this theorem depends on the work of Riemann [77], who nearly 40 years earlier, had the great insight to consider the variable $s$ to be a complex number in the Dirichet series and Euler product given in (1.2.1).

**Definition 1.2.1.** Let $s = \sigma + it$. The *Riemann zeta-function* is defined in the half-plane $\Re(s) > 1$ by either the Dirichlet series or the Euler product

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\substack{p \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The Riemann zeta-function is defined in the rest of the complex plane by analytic continuation except for a simple pole at $s = 1$. Let $\Gamma(n)$ denote the gamma function. For all $s \in \mathbb{C}$, the function

$$\Phi(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

satisfies the functional equation

$$\Phi(s) = \Phi(1-s).$$

By definition, $|\zeta(s)| > 0$ for $\sigma > 1$. Via the functional equation and well-known properties of $\Gamma(s)$, one can deduce that $\zeta(-2n) = 0$ for all natural numbers $n$. These are the so-called *trivial* zeros of $\zeta(s)$. Riemann showed that there are infinitely many *nontrivial* zeros of $\zeta(s)$, which are located in the critical strip, $0 \leq \sigma \leq 1$. He famously conjectured that all the nontrivial zeros of $\zeta(s)$ have real part equal to $1/2$. This statement is now called the Riemann Hypothesis, and it is considered to be one of the most important open problems in mathematics. Riemann found a deep connection between the nontrivial zeros of $\zeta(s)$ and the distribution of the primes. Indeed, the key to the analytic proof of the Prime Number Theorem is to show that $\zeta(1+it) \neq 0$. Moreover, the Riemann Hypothesis provides essentially the best possible bound for the error term in the Prime Number Theorem. (See [91].)

## 1.3 Generalizations of $\zeta(s)$

Let us now consider a generalization of $\zeta(s)$ by letting $K$ be a number field and $\mathcal{O}_K$ its ring of integers. In the half-plane $\Re(s) > 1$, the Dedekind zeta-function attached to $K$ is defined as

$$\zeta_K(s) := \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1},$$

where $I$ and $\mathfrak{p}$ run over the nonzero ideals and prime ideals of $\mathcal{O}_K$, respectively, and $N = N_{K/\mathbb{Q}}$ denotes the absolute norm on $K$.

The function $\zeta_K(s)$ extends meromorphically to the complex plane and has a simple pole at $s = 1$ with residue

$$\operatorname*{Res}_{s=1} \left\{\zeta_K(s)\right\} = \frac{2^{r_1}(2\pi)^{r_2}}{w|d|^{1/2}} hR. \tag{1.3.1}$$

Here $r_1$ denotes the number of real embeddings of $K$, $r_2$ denotes the number of pairs of complex embeddings of $K$, $w$ is the number of roots of unity, $d$ is the discriminant of $K$, $h$ is the class number of $K$, and $R$ is the regulator.

The function $\zeta_K(s)$ encodes information about the prime ideals of $\mathcal{O}_K$ due to the way in which unique factorization generalizes. In the case $K = \mathbb{Q}$, each element of $\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$ factors uniquely as the product of prime integers, a fact established by the Fundamental Theorem of Arithmetic. In this case, $\zeta_K(s) = \zeta(s)$, which we have seen encodes information about the prime integers. Such a factorization does not hold for other choices of $K$. For example, suppose $K = \mathbb{Q}[\sqrt{-5}]$. Then $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$, and we can see that all of the elements of $\mathbb{Z}[\sqrt{-5}]$ do not factor uniquely into irreducible elements of $\mathbb{Z}[\sqrt{-5}]$. For example,

$$6 = 2 \cdot 3 = (1+i\sqrt{5})(1-i\sqrt{5}),$$

and it is not difficult to show that $2, 3$, and $1 \pm i\sqrt{5}$ are irreducible. Dedekind made the discovery that the elements of $\mathcal{O}_K$ will, however, always factor uniquely into prime ideals. Thus the correct generalization of unique factorization in a number field $K$ is by way of prime ideals.

The Riemann zeta-function and Dedekend zeta-functions are members of a large class of functions, called $L$-functions. These functions can be defined in association with a plethora of mathematical objects, including Dirichlet characters, holomorphic cusp forms, and elliptic curves. Studying the analytic aspects of these functions is a worthwhile endeavor. As we have seen, the truth of the Prime Number Theorem depends upon the value of $\zeta(s)$ on the line $s = 1+it$. The residue of the pole of $\zeta_K(s)$, given in (1.3.1), encodes information about the class number of $K$. As a new example, the key component to the proof of Dirichlet's Theorem on primes in arithmetic progressions is the fact that the Dirichlet $L$-function $L(s, \chi)$ of a primitive character $\chi$ does not vanish at $s = 1$.

## 1.3 Properties of Automorphic $L$-functions on $\mathrm{GL}(m)$ over $\mathbb{Q}$

As a whole, the Langlands program predicts that the most general $L$-functions are attached to automorphic representations of $\mathrm{GL}(n)$ over a number field and further conjectures that these $L$-functions should be expressible as products of the Riemann zeta-function and automorphic $L$-functions attached to cuspidal automorphic representations of $\mathrm{GL}(m)$ over the rationals. We study the properties of such $L$-functions.

Let $\pi$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}(m)$ over $\mathbb{Q}$ with unitary central character. As before, let $s = \sigma + it$. For $\Re(s) > 1$, we let

$$L(s,\pi) := \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_{\substack{p \\ p \text{ prime}}} \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \tag{1.3.2}$$

be the global $L$-function attached to $\pi$ (as defined by Godement and Jacquet in [32] and Jacquet and Shalika in [51]). Here $m \in \mathbb{N}$ is called the *degree* of the $L$-function, and $\{\alpha_1(p), \ldots, \alpha_m(p)\}$ is the set of *local parameters* of the $L$-function. Furthermore, $a_\pi(1) = 1$, and $a_\pi(n), \alpha_j(p) \in \mathbb{C}$ for all $\pi, n, j$, and $p$.

An $L$-function is called *primitive* if it is not the product of two $L$-functions of smaller degree. The (primitive) function $L(s,\pi)$ is either the Riemann zeta-function or continues analytically to an entire function of order 1 satisfying a functional equation of the form

$$\begin{aligned} \Phi(s,\pi) &:= N^{s/2}\,\gamma(s,\pi)\,L(s,\pi) \\ &= \epsilon_\pi\,\overline{\Phi}(1-s,\pi), \end{aligned}$$

where $N$ is a natural number, $|\epsilon_\pi| = 1$, $\overline{\Phi}(s,\pi) = \overline{\Phi(\bar{s},\pi)}$, and the gamma factor

$$\gamma(s,\pi) = \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s+\mu_j).$$

Here $\Gamma_{\mathbb{R}}(s) = \pi^{s/2}\,\Gamma(s/2)$, and the $\mu_j$ are complex numbers. The Generalized Riemann Hypothesis states that all the nontrivial zeros of $L(s,\pi)$ are on the critical line $\Re(s) = 1/2$.

Logarithmically differentiating the Euler product given in 1.3.2, we define

$$-\frac{L'}{L}(s,\pi) := -\frac{d}{ds}\log L(s,\pi) = \sum_{p^\ell,\,\ell\geq 1} \frac{\left(\alpha_1^\ell(p)+\cdots+\alpha_m^\ell(p)\right)\log p}{p^{\ell s}} = \sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s}$$

for $\Re(s) > 1$. We note here that $\Lambda_\pi(p) = a_\pi(p)\log p$ for primes $p$. For an in-depth discussion of the theory of the $L$-functions, we refer the reader to Chapter 5 of the book by Iwaniec and Kowalski [50].

## 1.3 Properties of Twisted Automorphic $L$-functions on GL($m$) over $\mathbb{Q}$

We will also study the properties of automorphic $L$-functions on GL($m$) over $\mathbb{Q}$ twisted by Dirichlet characters. Let $\chi$ be a primitive Dirichlet character modulo $q$ satisfying $(q, N) = 1$, and let

$$L(s,\pi\otimes\chi) := \sum_{n=1}^{\infty}\frac{a_\pi(n)\chi(n)}{n^s} = \prod_{\substack{p \\ p \text{ prime}}}\prod_{j=1}^{m}\left(1-\frac{\alpha_j(p)\chi(p)}{p^s}\right)^{-1}$$

for $\Re(s) > 1$. Then

$$-\frac{L'}{L}(s,\pi\otimes\chi) := -\frac{d}{ds}\log L(s,\pi\otimes\chi) = \sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)\chi(n)}{n^s},$$

when $\Re(s) > 1$. For $q > 1$, the function $L(s,\pi\otimes\chi)$ continues to an entire function of order 1 and satisfies a functional equation of the form

$$\begin{aligned}\Phi(s,\pi\otimes\chi) &:= \left(q^m N\right)^{s/2}\gamma_\chi(s,\pi)\,L(s,\pi\otimes\chi)\\ &= \epsilon_{\pi,\chi}\,\overline{\Phi}(1-s,\pi\otimes\chi),\end{aligned}$$

where $|\epsilon_{\pi,\chi}| = 1$, $\overline{\Phi}(s,\pi\otimes\chi) = \overline{\Phi(\bar{s},\pi\otimes\chi)}$, and the gamma factor

$$\gamma_\chi(s,\pi) = \prod_{j=1}^{m}\Gamma_{\mathbb{R}}\left(s+\mu_{j,\chi}\right)$$

for complex numbers $\mu_{j,\chi}$. The Generalized Riemann Hypothesis states that all the nontrivial zeros of $L(s, \pi \otimes \chi)$ are on the critical line $\Re(s) = 1/2$.

## 1.4 Notation

Throughout this thesis, let $p$ denote a prime integer. We will make use of Landau's big-$O$ notation, $f(T) = O(g(T))$, and Vinogradov's notation, $f(T) \ll g(T)$, to mean that there exists a positive constant $C$ such that the inequality

$$|f(T)| \leq C|g(T)|$$

holds as $T \to \infty$. Unless otherwise stated, all constants implied by the big-$O$ or $\ll$ notations are absolute. We also use the expression $f(T) \gg g(T)$ to mean that

$$|f(T)| \geq C|g(T)|$$

as $T \to \infty$ where the implied constant is absolute. Finally, the notation $f(T) \sim g(T)$ as $T \to \infty$ means that

$$\lim_{T \to \infty} \frac{f(T)}{g(T)} = 1.$$

## 1.5 Continuous Moments of $L$-functions

The growth of a function and the distribution of its zeros are intimately connected, a relationship illustrated by the following theorem from complex analysis.

**Jensen's Formula.** *Let $f(z)$ be analytic for $|z| \leq R$, and suppose that $f(0) \neq 0$. If $\rho_1, \rho_2, \ldots, \rho_n$ are the zeros of $f(z)$ inside $|z| \leq R$, then*

$$\sum_{k=1}^{n} \log|\rho_k| = \log(|f(0)|R^n) - \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(Re^{i\theta})\right| \, d\theta.$$

An analogue of this formula for rectangles is very useful when working with Dirichlet series.

**Littlewood's Lemma.** *Let $f(s)$ be analytic and nonzero on the rectangle $\mathcal{C}$ with vertices $\sigma_0$, $\sigma_1$, $\sigma_1 + iT$, and $\sigma_0 + iT$, where $\sigma_0 < \sigma_1$. Then*

$$2\pi \sum_{\rho \in \mathcal{C}} Dist(\rho) = \int_0^T \log|f(\sigma_0+it)|\,dt - \int_0^T \log|f(\sigma_1+it)|\,dt + \int_{\sigma_0}^{\sigma_1} \arg f(\sigma_0+iT)\,d\sigma - \int_{\sigma_0}^{\sigma_1} \arg f(\sigma)\,d\sigma,$$

*where the sum runs over the zeros $\rho$ of $f(s)$ in $\mathcal{C}$ and $Dist(\rho)$ is the distance from $\rho$ to the left edge of the rectangle.*

By the arithmetic-geometric mean inequality, we have

$$\int_0^T \log|f(\sigma_0+it)|\,dt = \frac{1}{2k}\int_0^T \log|f(\sigma_0+it)|^{2k}\,dt \le \frac{T}{2k}\log\left(\frac{1}{T}\int_0^T |f(\sigma_0+it)|^{2k}\,dt\right),$$

which gives the connection between the mean-value estimate

$$\int_0^T |f(\sigma_0+it)|^{2k}\,dt$$

and the distance between certain zeros of $f(s)$ and the line $\Re(s) = \sigma_0$.

## 1.5 Moments of $\zeta(s)$

**Definition 1.5.1.** The $2k$th *moment* of the modulus of the Riemann zeta-function is defined as

$$I_k(T) := \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt,$$

where $k$ is any positive real number.

Much thought has been given to understanding $I_k(T)$ for different values of $k$, however finding an asymptotic expression of $I_k(T)$ for all $k > 0$ has proven to be a very difficult question. In 1918, Hardy and Littlewood [42] showed that

$$I_1(T) \sim T \log T$$

as $T \to \infty$. In 1926, Ingham [48] showed that

$$I_2(T) \sim \frac{T}{2\pi^2}(\log T)^4$$

as $T \to \infty$. No other asymptotic estimate has been proven for any other value of $k > 0$. Conjecturally,

$$I_k(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \sim c_k T(\log T)^{k^2}$$

for all $k > 0$, where $c_k$ is some constant depending on $k$. Obtaining such an asymptotic expression currently seems out of reach, however precise values of the constants $c_k$ have been conjectured using various techniques and approaches. For example, using number theoretic techniques, Conrey and Ghosh [18] conjectured the precise value of $c_3$, and Conrey and Gonek [24] conjectured the precise value of $c_4$. In 2000, Keating and Snaith [54] used techniques from random matrix theory to conjecture the constants $c_k$ for $k > -1/2$. In 2003, Diaconu, Goldfeld, and Hoffstein [27] used multiple Dirichlet series to obtain the constant conjectures for all $k \in \mathbb{N}$, as did Conrey, Farmer, Keating, Rubinstein, and Snaith [17] in 2006 using their $L$-function "recipe" and random matrix theory.

## 1.5 Gaps Between Zeros of Zeta-functions

In this section, we introduce Theorem 1.5.2, which pertains to the vertical distribution of nontrivial zeros $\zeta_K(s)$, where $K$ is a quadratic number field. This result is proved in Chapter 3 using the mixed second moments of derivatives of $\zeta_K(s)$ on the critical line. (See [89].)

First, we sketch the history of the problem for $\zeta(s)$. Denote the nontrivial zeros of $\zeta(s)$ as $\rho = \beta' + i\gamma'$, where $\beta', \gamma' \in \mathbb{R}$. It is known that for large $T$, the number of nontrivial zeros of $\zeta(s)$ up to height $T$ is

$$N(T) := \sum_{0<\gamma' \le T} 1 \;=\; \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Consider the sequence $0 < \gamma_1' \le \gamma_2' \le \dots$ of consecutive ordinates of the nontrivial zeros of $\zeta(s)$, and note that the average size of $\gamma_{n+1}' - \gamma_n'$ is $2\pi / \log \gamma_n'$. Normalizing, let

$$\lambda := \limsup_{n\to\infty} \frac{(\gamma_{n+1}' - \gamma_n') \log \gamma_n'}{2\pi}$$

and

$$\mu := \liminf_{n\to\infty} \frac{(\gamma_{n+1}' - \gamma_n') \log \gamma_n'}{2\pi}.$$

By definition, $\mu \le 1 \le \lambda$, but it is conjectured that $\mu = 0$ and $\lambda = \infty$. (See [65].) In other words, it is expected that there are arbitrarily small and large (normalized) gaps between consecutive nontrivial zeros of the Riemann zeta-function. Selberg (unpublished but announced in [81]) proved that $\mu < 1 < \lambda$. There is an abundance of quantitative results on the sizes of $\mu$ and $\lambda$, both unconditional and assuming various unproved hypotheses. See, for instance, [6], [8], [9], [10], [20], [21], [19], [30], [34], [40], [41], [65], [66], [72], [84], and [93]. Assuming the Riemann Hypothesis, the best current bounds are $\lambda \ge 2.9$ by Bui [9] and $\mu \le 0.5154$ by Feng and Wu [30].

Let us now consider the problem in a different setting. Let $K$ be a quadratic number field with discriminant $d$, and let $\chi_d$ be the Kronecker symbol of $d$. Then the Dedekind zeta-function factors as

$$\zeta_K(s) = \zeta(s) L(s, \chi_d), \tag{1.5.1}$$

where $\zeta(s)$ is the Riemann zeta-function and $L(s, \chi_d)$ is the Dirichlet $L$-function associated to $\chi_d$.

By understanding the moments of $\zeta_K(s)$ on the critical line, we can study the vertical distribution of the zeros of $\zeta_K(s)$ in the critical strip, which we denote by $\rho_K = \beta + i\gamma$. It has been shown that for an imaginary quadratic number field $K$, the vertical distribution of the nontrivial zeros of $\zeta_K(s)$ is related to the existence or non-existence of Landau-Siegel zeros and hence the size of the class number of $K$. This correspondence is described in the work of Conrey and Iwaniec [15]; see also Montgomery and Weinberger [68]. This circle of ideas is often referred to as the Deuring-Heilbronn phenomenon. For a very nice overview of the Deuring-Heilbronn phenomenon and its implications, see Stopple's survey article [87].

For a real or imaginary quadratic number field of discriminant $d$, it is known [50, Theorem 5.31] that for $T \geq 2$, we have

$$N_K(T) := \sum_{0<\gamma\leq T} 1 = \frac{T}{\pi} \log \frac{\sqrt{|d|}T}{(2\pi e)^2} + O\left(\log(\sqrt{|d|}T)\right).$$

Consider the sequence $0 < \gamma_1 \leq \gamma_2 \leq \ldots$ of consecutive ordinates of the nontrivial zeros of $\zeta_K(s)$, and note that the average size of $\gamma_{n+1} - \gamma_n$ is $\pi/\log(\sqrt{|d|}\gamma_n)$. Normalizing, let

$$\mu_K := \liminf_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\pi/\log(\sqrt{|d|}\gamma_n)}$$

and

$$\lambda_K := \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\pi/\log(\sqrt{|d|}\gamma_n)}.$$

By definition we have $\mu_K \leq 1 \leq \lambda_K$, however it is conjectured that $\mu_K = 0$ and $\lambda_K = \infty$. In other words, we expect that there are arbitrarily small and large normalized gaps between consecutive nontrivial zeros of Dedekind zeta-functions of quadratic number fields. While we expect $\mu_K = 0$, this is not due to the presumption of coincident nontrivial zeros of $\zeta(s)$ and $L(s, \chi_d)$. On the contrary, we expect that the zeros of $\zeta_K(s)$ are simple. Conrey, Ghosh, and Gonek [22] have shown that the number of simple zeros of $\zeta_K(s)$ with $0 < \gamma \leq T$ exceeds $T^{6/11}$ for sufficiently large $T$. In [23], the same authors show, assuming the Generalized

Riemann Hypothesis for Dirichlet $L$-functions, that a positive proportion of the zeros of $\zeta_K(s)$ are simple. In general, it is conjectured that any two distinct primitive $L$-functions should have no shared zero.

That $\mu_K < 1 < \lambda_K$ is an open question, and in particular there do not seem to be any quantitative results concerning the sizes of $\mu_K$ or $\lambda_K$. Towards finding a nontrivial lower bound for $\lambda_K$ , we prove the following unconditional theorem.

**Theorem 1.5.2.** *Let $T \geq 2$ and $\varepsilon > 0$. Let $K$ be a quadratic number field of discriminant $d$ with $|d| \leq T^{\frac{7}{9}-\varepsilon}$. There exists a subinterval of $[T, 2T]$ having length at least*

$$\sqrt{6} \cdot \frac{\pi}{\log \sqrt{|d|}T}\big(1 + O(|d|^{\varepsilon} \log^{-1} T)\big)$$

*for which the function $t \mapsto \zeta_K(1/2 + it)$ is free of zeros.*

Theorem 1.5.2 does not, *a fortiori*, state anything about the quantity $\lambda_K$. However, if we assume the Generalized Riemann Hypothesis for $\zeta_K(s)$, then Theorem 1.5.2 immediately implies the following inequality for $\lambda_K$.

**Corollary 1.5.3.** *Assume the Generalized Riemann Hypothesis for $\zeta_K(s)$. Then $\lambda_K \geq \sqrt{6}$. In particular, there are infinitely many normalized gaps between consecutive zeros of $\zeta_K(s)$ which are greater than $\sqrt{6} - \varepsilon$ times the average spacing for any $\varepsilon > 0$.*

The constant $\sqrt{6}$ in Corollary 1.5.3 is larger than one might expect since the same method of proof applied to the Riemann zeta-function only exhibits gaps between nontrivial zeros of $\zeta(s)$ of size $\sqrt{3}$ times the average spacing. (See [40].) Moreover, in contrast to Theorem 1.5.2 and its corollary, establishing a nontrivial upper bound on $\mu_K$ seems to be more difficult due to the connection to the Deuring-Heilbronn phenomenon and the class number problem for imaginary quadratic fields mentioned above.

## 1.5 Moments of Products of Automorphic $L$-functions

In this section, assuming some standard conjectures, we give upper bounds on moments of arbitrary products of automorphic $L$-functions. This is joint work with Micah B. Milinovich. (See [63].)

In general, given a primitive automorphic $L$-function, $L(s,\pi)$, normalized so that $\Re(s) = 1/2$ is the critical line, it has been conjectured [17] that there exist constants $C(k,\pi)$ such that

$$\int_0^T |L(\tfrac{1}{2}+it,\pi)|^{2k}\,dt \sim C(k,\pi)\,T\,(\log T)^{k^2}$$

for any $k > 0$ as $T \to \infty$. For degree one $L$-functions, the Riemann zeta-function and Dirichlet $L$-functions, the conjecture is known to hold when $k$ is 1 or 2. For degree two $L$-functions, many cases of the conjecture have been established when $k = 1$. See, for instance, results of Good [35] and Zhang [95, 96]. For higher degree $L$-functions, and for higher values of $k$, the conjecture seems to be beyond the scope of current techniques.

It is expected that the values of distinct primitive $L$-functions on the critical line are uncorrelated. Therefore, given $r$ distinct primitive $L$-functions, $L(s,\pi_1),\ldots,L(s,\pi_r)$, normalized so that $\Re(s) = 1/2$ is the critical line, one might conjecture that for any $k_1,\ldots,k_r > 0$ we have

$$\int_0^T |L(\tfrac{1}{2}+it,\pi_1)|^{2k_1}\cdots|L(\tfrac{1}{2}+it,\pi_r)|^{2k_r}\,dt \sim C(\vec{\mathbf{k}},\vec{\pi})\,T\,(\log T)^{k_1^2+\cdots+k_r^2} \tag{1.5.2}$$

for some constant $C(\vec{\mathbf{k}},\vec{\pi})$ as $T \to \infty$ where $\vec{\mathbf{k}} = (k_1,\ldots,k_r)$ and $\vec{\pi} = (\pi_1,\ldots,\pi_r)$. The conjectural order of magnitude of the moments in (1.5.2) is consistent with the observation that the logarithms of distinct primitive $L$-functions on the critical line, $\log L(\frac{1}{2}+it,\pi_1)$ and $\log L(\frac{1}{2}+it,\pi_2)$, are (essentially) statistically independent if $\pi_1 \ncong \pi_2$ as $t$ varies under the assumption of Selberg's orthogonality conjectures[2] for the Dirichlet series coefficients of

[2] For automorphic $L$-functions, we state Selberg's orthogonality conjectures in Chapter 4.

$L(s,\pi_1)$ and $L(s,\pi_2)$. This statistical independence can be made precise; see, for instance, the work of Bombieri and Hejhal [5] and Selberg [82].

Corollary A of [85] states that for the Riemann zeta-function the inequality

$$T(\log T)^{k^2} \ll_k \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon}$$

holds for any $k>0$ and every $\varepsilon>0$ assuming the Riemann Hypothesis. The upper bound is due to Soundararajan, and the lower bound is due to Ramachandra [76]. In May of 2013, Harper [44] refined the ideas of Soundararajan and proved, under the assumption of the Riemann Hypothesis, that for every positive real number $k$, we have

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt \ll_k T(\log T)^{k^2},$$

where the indicated constant depends on $k$. We note that Harper uses Soundararajan's upper bounds for moments of $\zeta(s)$ in [85] to prove this result.

In support of the conjecture in (1.5.2), we have proven the following mean-value estimate for arbitrary products of primitive automorphic $L$-functions.

**Theorem 1.5.4.** *Let $L(s,\pi_1),\ldots,L(s,\pi_r)$ be $L$-functions attached to distinct irreducible cuspidal automorphic representations, $\pi_j$, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ each with unitary central character, and assume that these $L$-functions satisfy the Generalized Riemann Hypothesis. Then, if $\max\limits_{1\le j\le r} m_j \le 4$, we have*

$$\int_0^T |L(\tfrac{1}{2}+it,\pi_1)|^{2k_1}\cdots|L(\tfrac{1}{2}+it,\pi_r)|^{2k_r}\,dt \ll T(\log T)^{k_1^2+\cdots+k_r^2+\varepsilon} \tag{1.5.3}$$

*for any $k_1,\ldots,k_r>0$ and every $\varepsilon>0$ when $T$ is sufficiently large. The implied constant in* (1.5.3) *depends on $\pi_1,\ldots,\pi_r$, $k_1,\ldots,k_r$, and $\varepsilon$. If $\max\limits_{1\le j\le r} m_j \ge 5$, then the inequality in* (1.5.3) *holds under the additional assumption of Hypothesis H, which is given in Chapter 4.*

Observe that the upper bound in Theorem 1.5.4 is nearly as sharp as the conjectured asymptotic formula in (1.5.2). In the case $r = 1$, combining the result of Theorem 1.5.4 with the work of Pi [73], we deduce that

$$T(\log T)^{k^2} \ll_{k,\pi} \int_0^T |L(\tfrac{1}{2}+it,\pi)|^{2k}\,dt \ll_{\pi,k,\varepsilon} T(\log T)^{k^2+\varepsilon} \tag{1.5.4}$$

for any $k > 0$ and every $\varepsilon > 0$ where $\pi$ is a self-contragredient irreducible cuspidal automorphic representations of $\mathrm{GL}(m)$ over $\mathbb{Q}$ under the assumptions of the Generalized Riemann Hypothesis and the Ramanujan-Petersson Conjecture[3] for $L(s,\pi)$. The upper bound holds under weaker assumptions and for more general $L$-functions. We may let $L(s,\pi_1) = \zeta(s)$ in the proof of Theorem 1.5.4, so our theorem generalizes Soundararajan's result. As is the case in [85], it is possible to replace the $\varepsilon$ in Theorem 1.5.4 by a quantity which is $O(1/\log\log\log T)$; see Ivić [49]. Moreover, note that we do not assume that the $L$-functions in Theorem 1.5.4 satisfy the Ramanujan-Petersson Conjecture. Instead, we assume Hypothesis H of Rudnick and Sarnak [79]. This mild (but unproven) conjecture is implied by the Ramanujan-Petersson Conjecture and is known to hold for $L$-functions attached to irreducible cuspidal automorphic representations on $\mathrm{GL}(m)$ over $\mathbb{Q}$ if $m \leq 4$.

Finally we remark that, assuming the Generalized Riemann Hypothesis and the Ramanujan-Petersson Conjecture, Pi [73] has shown that the integral in (1.5.4) is $\ll T(\log T)^{k^2}$ if $\pi$ is self-contragredient for any fixed $k$ satisfying $0 < k < 2/m$. Moreover, lower bounds for the integral in (1.5.4) which are $\gg T(\log T)^{k^2}$ for any positive rational number $k$ have been established by Akbary and Fodden [1] assuming unproven bounds toward the Ramanujan-Petersson Conjecture but without assuming the Generalized Riemann Hypothesis. The results in [1] are unconditional in the case $m = 2$.

[3]For automorphic $L$-functions, we state the Ramanujan-Petersson Conjecture in Chapter 4.

## 1.5 Moments of Dedekind zeta-functions

Let $K$ be an algebraic number field. It is known that the Dedekind zeta-function attached to $K$ factors as a product of Artin $L$-functions. For instance, if $K$ is a Galois extension of $\mathbb{Q}$, then

$$\zeta_K(s) = \prod_{\chi} L(s,\chi)^{\chi(1)} \tag{1.5.5}$$

where the product is over the irreducible characters $\chi$ of $\mathrm{Gal}(K/\mathbb{Q})$ and

$$\sum_{\chi} \chi(1)^2 = |\mathrm{Gal}(K/\mathbb{Q})| = [K:\mathbb{Q}]. \tag{1.5.6}$$

The Langlands reciprocity conjecture implies that each $L(s,\chi) = L(s,\pi)$ for an irreducible cuspidal automorphic representation $\pi$ of $\mathrm{GL}(m)$ over $\mathbb{Q}$ where $\chi(1) = m$. By (1.5.2), (1.5.5), and (1.5.6), for Galois extensions $K$ over $\mathbb{Q}$, this leads to the conjecture that

$$\int_0^T \left|\zeta_K(\tfrac{1}{2}+it)\right|^{2k} dt \sim C(k,K)\, T\, (\log T)^{[K:\mathbb{Q}]k^2} \tag{1.5.7}$$

for any $k > 0$ as $T \to \infty$. Here $C(k,K)$ is a constant depending on $k$ and the number field $K$. The recent work of Heap [45] discusses this conjecture in more detail.

The conjectural asymptotic formula in (1.5.7) is known to hold when $k = 1$ for the Dedekind zeta-functions of quadratic extensions of $\mathbb{Q}$. Let $d$ be a fundamental discriminant, and let $K = \mathbb{Q}[\sqrt{d}]$. Then Motohashi [70] has shown that

$$\int_0^T \left|\zeta_K(\tfrac{1}{2}+it)\right|^{2} dt \sim \frac{6}{\pi^2} L(1,\chi_d)^2 \prod_{p|d}\left(1+\frac{1}{p}\right)^{-1} T \log^2 T$$

as $T \to \infty$ using the factorization $\zeta_K(s) = \zeta(s)L(s,\chi_d)$, where $L(s,\chi_d)$ is the Dirichlet $L$-function associated to $\chi_d$, the Kronecker symbol of $d$. Also in support of (1.5.7), for finite

Galois extensions $K$ over $\mathbb{Q}$, Akbary and Fodden [1] have shown that the inequality

$$\int_0^T \left|\zeta_K(\tfrac{1}{2}+it)\right|^{2k} dt \gg T(\log T)^{[K:\mathbb{Q}]k^2}$$

holds for any rational number $k > 0$ as $T \to \infty$.

Using results of Arthur and Clozel [2], the following mean-value estimate for Dedekind zeta-functions is a consequence of Theorem 1.5.4.

**Corollary 1.5.5.** *Let $K$ be a finite solvable Galois extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the associated Dedekind zeta-function. Then, assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, we have*

$$\int_0^T \left|\zeta_K(\tfrac{1}{2}+it)\right|^{2k} dt \ll_{K,k,\varepsilon} T(\log T)^{[K:\mathbb{Q}]k^2+\varepsilon}$$

*for any $k, \varepsilon > 0$ when $T$ is sufficiently large.*

The condition that $\mathrm{Gal}(K/\mathbb{Q})$ be a solvable group can be removed by approaching the problem in a more algebraic way.

**Theorem 1.5.6.** *Let $K$ be a finite Galois extension of $\mathbb{Q}$. Then, assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, we have*

$$\int_0^T \left|\zeta_K(\tfrac{1}{2}+it)\right|^{2k} dt \ll_{K,k,\varepsilon} T(\log T)^{[K:\mathbb{Q}]k^2+\varepsilon}$$

*for any $k, \varepsilon > 0$ when $T$ is sufficiently large.*

Unlike the proof of Corollary 1.5.5, our proof of Theorem 1.5.6 does not rely on a factorization of $\zeta_K(s)$ into automorphic $L$-functions.

## 1.5 Coefficients of Zeta- and $L$-functions in Short Intervals

As an application of Theorem 1.5.6, let $K$ be a number field with discriminant $d$, and let $r_K(n)$ denote the number of ideals in $K$ of norm $n$. Then, by the definition of $\zeta_K(s)$, we see that

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{r_K(n)}{n^s}, \quad \Re(s) > 1.$$

When $K$ is a Galois extension of $\mathbb{Q}$, we can use Theorem 1.5.6 and a technique of Selberg [80] to study the distribution of $r_K(n)$ in short intervals assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$. In order to state our result, recall from (1.3.1) that

$$\operatorname*{Res}_{s=1} \big\{\zeta_K(s)\big\} \; = \; \lim_{s\to 1}(s-1)\zeta_K(s) \; = \; \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d|}}$$

where $r_1$ is the number of real embeddings of $K$, $r_2$ is the number of pairs of complex embeddings, $h$ is the class number of $K$, $R$ is the regulator, $w$ is the number of roots of unity in $K$, and $d$ is the discriminant. Landau's classical mean-value estimate for the arithmetic function $r_K(n)$ is

$$\sum_{n\le x} r_K(n) = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d|}}\, x + O\Big(x^{1-2/([K:\mathbb{Q}]+1)}\Big)\,.$$

In Chapter 6, we prove the following conditional estimate for the variance of the arithmetic function $r_K(n)$ in short intervals. This result, proved in collaboration with Micah B. Milinovich, appears in [63].

**Theorem 1.5.7.** *Let $K$ be a finite Galois extension of $\mathbb{Q}$. Let $y = y(x)$ be a positive and increasing function such that $y \to \infty$ and $y/x \to 0$ as $x \to \infty$. Then, assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, we have*

$$\frac{1}{X}\int_X^{2X} \Bigg| \sum_{x<n\le x+y} r_K(n) - \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d|}}\, y \Bigg|^2 dx \;\ll\; y\,(\log X)^{[K:\mathbb{Q}]+\varepsilon}$$

*for $\varepsilon > 0$ when $X$ is sufficiently large. Here the implied constant depends on $K$ and $\varepsilon$.*

Assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, it follows from Theorem 1.5.7 that

$$\sum_{x<n\le x+y} r_K(n) \;\sim\; \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{|d|}}\, y$$

for *almost all* $x$ if we choose $y$ to be a function of $x$ satisfying $y/(\log x)^{[K:\mathbb{Q}]+\varepsilon} \to \infty$ but $y/x \to 0$ as $x \to \infty$.

Using Theorem 1.5.4, we can similarly study the behavior of coefficients of products of automorphic $L$-functions in short intervals. To state the results in this situation, we first introduce some notation. For $k \geq 0$ an integer and $k_1, \ldots, k_r \in \mathbb{N}$, let

$$L(s) = \zeta(s)^k \prod_{j=1}^{r} L(s, \pi_j)^{k_j}$$

be an (automorphic) $L$-function. Here we are assuming that the $L$-functions $L(s, \pi_1), \ldots, L(s, \pi_r)$ are as in Theorem 1.5.4 and that $L(s, \pi_j) \neq \zeta(s)$ for all $1 \leq j \leq r$. We distinguish between the case $k = 0$, where $L(s)$ is entire, and the case $k \geq 1$, where $L(s)$ has a pole of order $k$ at $s = 1$. For $\Re(s) > 1$, we set

$$L(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{a_L(n)}{n^s}, & \text{if } k = 0, \\ \displaystyle\sum_{n=1}^{\infty} \frac{b_L(n)}{n^s}, & \text{if } k \in \mathbb{N}. \end{cases}$$

As is to be expected, the behavior of $a_L(n)$ and $b_L(n)$ in short intervals differs due to the presence of the pole of the generating function when $k \geq 1$. For $x > 0$, we define

$$R_L(x) = \operatorname*{Res}_{s=1} \left( L(s) \frac{x^s}{s} \right).$$

Note that $R_L(x) = 0$ if $k = 0$, that

$$R_L(x) \ = \ x \prod_{j=1}^{r} L(1, \pi_j)^{k_j}$$

if $k=1$, and that

$$R_L(x) \;=\; \frac{x(\log x)^{k-1}}{(k-1)!}\prod_{j=1}^{r} L(1,\pi_j)^{k_j} + O\big(x(\log x)^{k-2}\big)$$

if $k \geq 2$. In Chapter 6, we modify the proof of Theorem 1.5.7 to prove the following theorem.

**Theorem 1.5.8.** *Let $L(s,\pi_1),\ldots,L(s,\pi_r)$ be $L$-functions attached to distinct irreducible cuspidal automorphic representations, $\pi_j$, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ each with unitary central character, and assume that these $L$-functions satisfy the Generalized Riemann Hypothesis. Let $y=y(x)$ be a positive and increasing function such that $y \to \infty$ and $y/x \to 0$ as $x \to \infty$. Then, if $\max\limits_{1\leq j\leq r} m_j \leq 4$, we have*

$$\frac{1}{X}\int_X^{2X}\Bigg|\sum_{x<n\leq x+y} a_L(n)\Bigg|^2 dx \;\ll\; y\,(\log X)^{k_1^2+\cdots+k_r^2+\varepsilon}$$

*and*

$$\frac{1}{X}\int_X^{2X}\Bigg|\sum_{x<n\leq x+y} b_L(n) - \Big(R_L(x+y)-R_L(x)\Big)\Bigg|^2 dx \;\ll\; y\,(\log X)^{k^2+k_1^2+\cdots+k_r^2+\varepsilon}$$

*for $\varepsilon>0$ when $X$ is sufficiently large and the implied constants depend on $\pi_1,\ldots,\pi_r$, $k$, $k_1,\ldots,k_r$, and $\varepsilon$. If $\max\limits_{1\leq j\leq r} m_j \geq 5$, then the result holds under the additional assumption of Hypothesis H, which is given in Chapter 4.*

## 1.6 Moments of Quadratic Twists of $L$-functions

In this section, assuming the Generalized Riemann Hypothesis and the Ramanujan-Petersson Conjecture, we give upper bounds on moments of arbitrary products of automorphic $L$-functions twisted by quadratic Dirichlet characters. This is joint work with Micah B. Milinovich. (See [63].)

One can use the methods of Soundararajan in [85] to study the moments of central values of quadratic twists of automorphic $L$-functions. In this case, the conjecture for the size of moments depends on the symmetry type of the family of these twists. Let $L(s,\pi)$ be an $L$-function attached to a self-contragredient irreducible cuspidal automorphic representation $\pi$ on $\mathrm{GL}(m)$ over $\mathbb{Q}$. (We assume the $L$-function is self-dual so that the central value is real.) Then Katz and Sarnak [52] and Rubinstein [78] have conjectured that the family of quadratic twists of $L(s,\pi)$ has either symplectic or orthogonal symmetry corresponding to whether or not the symmetric square $L$-function $L(s,\pi,\wedge^2)$ has a pole at $s=1$.

Following the notation in [78], we set $\delta(\pi)=1$ if $L(s,\pi,\wedge^2)$ does not have a pole at $s=1$ and set $\delta(\pi)=-1$ if $L(s,\pi,\wedge^2)$ has a pole at $s=1$. Then for each $k>0$ it has been conjectured (see [16, 53]) that there are constants $C^\flat(k,\pi)>0$ such that

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\pi\otimes\chi_d)^k \sim C^\flat(k,\pi)X(\log X)^{k(k-\delta(\pi))/2}$$

as $X\to\infty$. Here the superscript $\flat$ indicates that the sums run over fundamental discriminants $d$, $\chi_d$ denotes the corresponding primitive quadratic Dirichlet character, and (as before) we have normalized so that $s=1/2$ is the central point. In the case of quadratic Dirichlet $L$-functions and $L$-functions of quadratic twists of a fixed elliptic curve $E$, Soundararajan [85] proved that

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\chi_d)^k \ll X(\log X)^{k(k+1)/2+\varepsilon} \tag{1.6.1}$$

and

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},E\otimes\chi_d)^k \ll X(\log X)^{k(k-1)/2+\varepsilon} \tag{1.6.2}$$

for every $k>0$ and any $\varepsilon>0$ assuming the Generalized Riemann Hypothesis for the relevant $L$-functions. (Note that in the first example the $L$-functions have $\delta(\pi)=-1$, and in the second case the $L$-functions have $\delta(\pi)=1$.)

## 1.6 Moments of Products of Quadratic Twists of Automorphic $L$-functions

We generalize the above results of Soundararajan and, in analogy with our Theorem 1.5.4, we prove the following result for central values of quadratic twists of arbitrary products of automorphic $L$-functions.

**Theorem 1.6.1.** *Let $d$ denote a fundamental discriminant, and let $\chi_d$ be a primitive quadratic Dirichlet character of conductor $|d|$. Let $L(s,\pi_1),\ldots,L(s,\pi_r)$ be $L$-functions attached to distinct self-contragredient irreducible cuspidal automorphic representations, $\pi_j$, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ each with unitary central character, and assume that the twisted $L$-functions $L(s,\pi_1\otimes\chi_d),\ldots,L(s,\pi_r\otimes\chi_d)$ satisfy the Generalized Riemann Hypothesis and the Ramanujan-Petersson Conjecture. Then we have*

$$\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\pi_1\otimes\chi_d)^{k_1}\cdots L(\tfrac{1}{2},\pi_r\otimes\chi_d)^{k_r} \ll X(\log X)^{k_1(k_1-\delta(\pi_1))/2+\cdots+k_r(k_r-\delta(\pi_r))/2+\varepsilon}, \quad (1.6.3)$$

*for any $k_1,\ldots,k_r>0$ and every $\varepsilon>0$ when $X$ is sufficiently large. Here the superscript $\flat$ indicates that the sum is restricted to fundamental discriminants, and the implied constant depends on $\pi_1,\ldots,\pi_r$, $k_1,\ldots,k_r$, and $\varepsilon$.*

We now give two examples which are consequences of Theorem 1.6.1 and generalize Soundararajan's results in (1.6.1) and (1.6.2) to biquadratic extensions of $\mathbb{Q}$. Let $d_1$ and $d_2$ be coprime fundamental discriminants, and let $K_{d_1,d_2}=\mathbb{Q}[\sqrt{d_1},\sqrt{d_2}]$ be the corresponding biquadratic number field. Then the Dedekind zeta-function of $K_{d_1,d_2}$ factors as

$$\zeta_{K_{d_1,d_2}}(s)=\zeta(s)L(s,\chi_{d_1})L(s,\chi_{d_2})L(s,\chi_{d_1d_2}),$$

and similarly, given an elliptic curve $E$ over $\mathbb{Q}$, the Hasse-Weil $L$-function $L(s,E/K_{d_1,d_2})$ of $E$ over $K_{d_1,d_2}$ factors as

$$L(s,E/K_{d_1,d_2})=L(s,E)L(s,E\otimes\chi_{d_1})L(s,E\otimes\chi_{d_2})L(s,E\otimes\chi_{d_1d_2}).$$

Using Theorem 1.6.1, we can estimate moments of $\zeta_{K_{d_1,d_2}}(\frac{1}{2})$ and $L(\frac{1}{2}, E/K_{d_1,d_2})$ by averaging over two sets of fundamental discriminants. (We note that under the assumption of the Generalized Riemann Hypothesis for these zeta- and $L$-functions, these central values are non-negative real numbers.) In particular, we have

$$\sum_{\substack{|d_1 d_2| \le X \\ (d_1,d_2)=1}}^{\flat} \zeta_{K_{d_1,d_2}}(\tfrac{1}{2})^k \ \ll\ X\,(\log X)^{3k(k+1)/2+1+\varepsilon} \tag{1.6.4}$$

and

$$\sum_{\substack{|d_1 d_2| \le X \\ (d_1,d_2)=1}}^{\flat} L(\tfrac{1}{2}, E/K_{d_1,d_2})^k \ \ll\ X\,(\log X)^{3k(k-1)/2+1+\varepsilon} \tag{1.6.5}$$

for any $\varepsilon > 0$. Here the superscript $\flat$ indicates that the sum runs over two sets fundamental discriminants, $d_1$ and $d_2$. When $k = 1$, the conditional estimate in (1.6.4) is consistent with a result of Chinta [14] who proved that, as $X \to \infty$,

$$\sum_{d_1,d_2 \text{ odd}}^{\flat} a(d_1,d_2)\zeta_{K_{d_1,d_2}}(\tfrac{1}{2})F\left(\frac{d_1 d_2}{X}\right) \ \sim\ cX\log^4 X$$

for a constant $c > 0$, where $F$ is a smooth compactly supported test function satisfying $\int_0^\infty F(x)\,dx = 1$ and $a(d_1,d_2)$ is a weighting factor satisfying $a(d_1,d_2) = 1$ if $(d_1,d_2) = 1$ and is (on average) small otherwise.

Since the condition $(d_1,d_2) = 1$ implies that $\chi_{d_1 d_2} = \chi_{d_1}\chi_{d_2}$, and $\delta(\pi) = -1$ for any degree one $L$-function, under the conditions of Theorem 1.6.1 we have

$$\begin{aligned}
\sum_{\substack{|d_1 d_2| \le X \\ (d_1,d_2)=1}}^{\flat} \zeta_{K_{d_1,d_2}}(\tfrac{1}{2})^k \ &=\ \zeta(\tfrac{1}{2})^k \sum_{|d_1| \le X}^{\flat} L(\tfrac{1}{2},\chi_{d_1})^k \sum_{\substack{|d_2| \le X/|d_1| \\ (d_1,d_2)=1}}^{\flat} L(\tfrac{1}{2},\chi_{d_2})^k L(\tfrac{1}{2},\chi_{d_1 d_2})^k \\
&\ll\ X(\log X)^{k(k+1)+\varepsilon} \sum_{|d_1| \le X}^{\flat} \frac{L(\frac{1}{2},\chi_{d_1})^k}{|d_1|} \\
&\ll\ X(\log X)^{3k(k+1)/2+1+\varepsilon}
\end{aligned}$$

by two applications of (1.6.3) and summation by parts. This proves that the estimate in (1.6.4) follows from Theorem 1.6.1.

To prove (1.6.5), we observe that the modularity theorems of Wiles [92], Wiles and Taylor [88], and Breuil, Conrad, Diamond, and Taylor [7] imply that $L(s,E)$ and its quadratic twists correspond to $L$-functions attached to irreducible cuspidal automorphic representations of GL(2) over $\mathbb{Q}$. Moreover, we have $\delta(\pi)=1$ for each of these $L$-functions. Therefore, under the conditions of Theorem 1.6.1, we similarly have

$$\begin{aligned}\sideset{}{^\flat}\sum_{\substack{|d_1d_2|\le X\\(d_1,d_2)=1}} L(\tfrac12,E/K_{d_1,d_2})^k &= L(\tfrac12,E)^k \sideset{}{^\flat}\sum_{|d_1|\le X} L(\tfrac12,E\otimes\chi_{d_1})^k \sideset{}{^\flat}\sum_{\substack{|d_2|\le X/|d_1|\\(d_1,d_2)=1}} L(\tfrac12,E\otimes\chi_{d_2})^k L(\tfrac12,E\otimes\chi_{d_1d_2})^k\\ &\ll X(\log X)^{k(k-1)+\varepsilon} \sideset{}{^\flat}\sum_{|d_1|\le X} \frac{L(\tfrac12,E\otimes\chi_{d_1})^k}{|d_1|}\\ &\ll X(\log X)^{3k(k-1)/2+1+\varepsilon}\end{aligned}$$

by two more applications of (1.6.3) and summation by parts. This shows that the estimate in (1.6.5) also follows from Theorem 1.6.1.

# 2 THE PROOF OF THEOREM 1.1.2 AND ITS COROLLARIES

In this chapter, we prove Theorem 1.1.2, that is, we establish the existence of $m$-tuples that infinitely often represent strings of consecutive prime numbers. The proof is based on the recent work of Maynard [62] and Tao which proves the existence of $m$-tuples that infinitely often represent strings of prime numbers. We also give proofs to Corollary 1.1.3, Corollary 1.1.4, and Corollary 1.1.5. The results in this chapter were proved jointly with William D. Banks and Tristan Freiberg. The proofs given here are slightly expanded versions of the proofs that appear in our article [4].

## 2.1 The Proof of Theorem 1.1.2

We now prove Theorem 1.1.2.

*Proof.* Let $m, k \in \mathbb{N}$ with $m \geq 2$ and $k \geq k_m$, where $k_m \log k_m > \mathrm{e}^{8m+4}$. Let $b_1, \ldots, b_k$ be distinct integers such that $\{x+b_j\}_{j=1}^{k}$ is admissible, and let $g$ be any positive integer coprime with $b_1 \cdots b_k$. Notice that, for any integer $B$, the $k$-tuple

$$\{x + b_j + gB\}_{j=1}^{k}$$

is also admissible. Thus we may assume, without loss of generality, that

$$1 < b_1 < \cdots < b_k.$$

We will now construct a new admissible $k$-tuple of linear forms which will generate strings of consecutive primes infinitely often. Let $r = b_k - k \geq 1$, and choose arbitrary primes $q_1 < \cdots < q_r$ coprime to $g$. For each $q_i$, we have $(g, q_i) = 1$, and thus the linear congruence

$$ga_t + t \equiv 0 \bmod q_t$$

has a solution $a_t$, say. By the Chinese Remainder Theorem, we can find an integer $a$ such that

$$ga + t \equiv 0 \bmod q_t \qquad (1 \leq t \leq b_k \quad \text{and} \quad t \notin \{b_1, \ldots, b_k\}).$$

Consider the $k$-tuple

$$\mathcal{A}(x) = \{gQx + ga + b_j\}_{j=1}^{k}$$

where $Q := q_1 \cdots q_r$. Since $\{x + b_j\}_{j=1}^{k}$ is admissible and $t \notin \{b_1, \ldots, b_k\}$, it follows that $\mathcal{A}(x)$ is also admissible. Moreover, $\mathcal{A}(x)$ satisfies (1.1.1) (with $g_j = gQ$ and $h_j = ga + b_j$) since the integers $b_1, \ldots, b_k$ are distinct and $gQ \geq 1$. For every $N \in \mathbb{N}$ our choices of $Q$ and $a$ guarantee that

$$g(QN + a) + t \equiv 0 \bmod q_t \qquad (1 \leq t \leq b_k \quad \text{and} \quad t \notin \{b_1, \ldots, b_k\}).$$

Consequently, any prime number in the interval

$$\Big[g(QN + a) + b_1, \;\; g(QN + a) + b_k\Big]$$

must lie in $\mathcal{A}(N)$.

Now let $m'$ be the largest integer such that there is a subset $\{h_1, \ldots, h_{m'}\}$ of $\{b_1, \ldots, b_k\}$ with the property that the $m'$ integers

$$g(QN + a) + h_i \qquad (1 \leq i \leq m') \tag{2.1.1}$$

are simultaneously prime for infinitely many $N \in \mathbb{N}$. Since $k \geq k_m$ we can apply the Maynard–Tao Theorem with $\mathcal{A}(x)$ to deduce that $m' \geq m$.

By the maximal property of $m'$, it must be the case that for all sufficiently large $N \in \mathbb{N}$, if the numbers in (2.1.1) are all prime, then $g(QN + a) + b_j$ is composite for every $b_j \in \{b_1, \ldots, b_k\} \setminus \{h_1, \ldots, h_{m'}\}$. Hence, for infinitely many $N \in \mathbb{N}$, the interval

$$\Big[g(QN + a) + b_1, \;\; g(QN + a) + b_k\Big]$$

contains precisely $m'$ consecutive primes, namely, the numbers

$$\{gn + h_i\}_{i=1}^{m'}$$

with $n = QN + a$. This completes the proof of Theorem 1.1.2. $\square$

## 2.2 The Proof of Corollary 1.1.3

In this section, we prove Corollary 1.1.3, which in particular, answers the question of Erdős and Turán [29] given in Section 1.1.1.

*Proof.* Let $m \geq 2$, and let $k$ be sufficiently large in terms of $m$. Let $\mathcal{B}(x) = \{x + 2^j\}_{j=1}^{k}$, which is easily seen to be admissible. By Theorem 1.1.2, there exists a tuple

$$\mathcal{H}(x) = \{x + 2^{\nu_j}\}_{j=1}^{m+1} \subseteq \mathcal{B}(x)$$

such that $\mathcal{H}(n)$ is an $(m + 1)$-tuple of consecutive primes for infinitely many $n$. Here, $1 \leq \nu_1 < \cdots < \nu_{m+1} \leq k$. For such $n$, writing

$$\mathcal{H}(n) = \{n + 2^{\nu_j}\}_{j=1}^{m+1} = \{p_{r+1}, \ldots, p_{r+m+1}\}$$

with some integer $r$, we have

$$\delta_j = d_{r+j} = p_{r+j+1} - p_{r+j} = 2^{\nu_{j+1}} - 2^{\nu_j}$$

for $1 \le j \le m$. Then

$$\begin{aligned}\sum_{i=1}^{j-1} \delta_i &= \sum_{i=1}^{j-1} (2^{\nu_{i+1}} - 2^{\nu_i}) \\ &= 2^{\nu_j} - 2^{\nu_1} \\ &< 2^{\nu_{j+1}} - 2^{\nu_j} \\ &= \delta_j\end{aligned}$$

for $2 \le j \le m$. Hence,

$$\delta_{j-1} \le \delta_1 + \cdots + \delta_{j-1} < \delta_j$$

for each $j$, which proves the first statement. To obtain runs of consecutive prime gaps with

$$\delta_j > \delta_{j+1} + \cdots + \delta_m \ge \delta_{j+1},$$

consider instead the admissible $k$-tuple $\{x - 2^j\}_{j=1}^k$. This completes the proof. □

## 2.3 The Proof of Corollary 1.1.4

In this section, we prove that for every natural number $m \ge 2$, there are infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps such that $\delta_{j-1} \mid \delta_j$ for $2 \le j \le m$ and infinitely many runs such that $\delta_{j+1} \mid \delta_j$ for $1 \le j \le m - 1$.

*Proof.* Let $m \geq 2$, let $k$ be sufficiently large in terms of $m$. Put $Q := \prod_{p \leq k} p$, and define the sequence $b_1, \ldots, b_k$ inductively as follows. Let

$$b_1 = 0, \quad b_2 = Q, \quad b_3 = 2Q,$$

and for any $j \geq 3$, let

$$b_j = b_{j-1} + \prod_{1 \leq s < t \leq j-1} (b_t - b_s).$$

Note that, for $v \geq u \geq 1$, we have

$$(b_{u+1} - b_u) \mid (b_{v+1} - b_v). \tag{2.3.1}$$

Now let $\mathcal{B}(x) = \{x + b_j\}_{j=1}^{k}$, and observe that $\mathcal{B}(x)$ is admissible since $Q$ divides each integer $b_j$. By Theorem 1.1.2, there exists a tuple

$$\mathcal{H}(x) = \{x + b_{\nu_j}\}_{j=1}^{m+1} \subseteq \mathcal{B}(x)$$

such that $\mathcal{H}(n)$ is an $(m+1)$-tuple of consecutive primes for infinitely many $n$. Here, as before, $1 \leq \nu_1 < \cdots < \nu_{m+1} \leq k$. For any such $n$, writing

$$\mathcal{H}(n) = \{n + b_{\nu_j}\}_{j=1}^{m+1} = \{p_{r+1}, \ldots, p_{r+m+1}\}$$

with some integer $r$, we have

$$\delta_j = d_{r+j} = p_{r+j+1} - p_{r+j} = b_{\nu_{j+1}} - b_{\nu_j}$$

for $1 \leq j \leq m$. Then

$$\prod_{i=1}^{j-1} \delta_i = \prod_{i=1}^{j-1} (b_{\nu_{i+1}} - b_{\nu_i}) \Bigg| \prod_{1 \leq s < t \leq \nu_j} (b_t - b_s) = b_{\nu_j+1} - b_{\nu_j}$$

if $2 \le j \le m$. On the other hand, using (2.3.1) we see that

$$(b_{\nu_j+1} - b_{\nu_j}) \Bigg| \sum_{i=\nu_j}^{\nu_{j+1}-1} (b_{i+1} - b_i) = b_{\nu_{j+1}} - b_{\nu_j} = \delta_j.$$

Hence, $\delta_1 \cdots \delta_{j-1} \mid \delta_j$ for $2 \le j \le m$, which proves the first statement. To obtain runs of consecutive prime gaps with

$$\delta_m \delta_{m-1} \cdots \delta_{j+1} \mid \delta_j$$

for $1 \le j \le m-1$, consider instead the admissible $k$-tuple $\{x - b_j\}_{j=1}^k$. $\square$

## 2.4 The Proof of Corollary 1.1.5

In this section, we prove a strengthening of Shiu's Theorem.

*Proof.* Let $m \ge 2$, let $k$ be sufficiently large in terms of $m$. Since $(a, D) = 1$, there are infinitely many primes in the arithmetic progression $a$ mod $D$. Let $q_1 < \cdots < q_k$ be primes congruent to $a$ mod $D$, with $q_1 > k$. Then $\mathcal{B}(x) = \{x + q_j\}_{j=1}^k$ is admissible. By Theorem 1.1.2, we deduce that there is some

$$\mathcal{H}(x) = \{x + h_j\}_{j=1}^m \subseteq \mathcal{B}(x)$$

such that, for infinitely many $n \in \mathbb{N}$, $\mathcal{H}(Dn)$ is an $m$-tuple of consecutive primes. These consecutive primes lie in the arithmetic progression $a$ mod $D$ and are contained in an interval of length

$$(Dn + h_m) - (Dn + h_1) = h_m - h_1,$$

as desired. $\square$

# 3 THE PROOF OF THEOREM 1.5.2

In this chapter, we prove Theorem 1.5.2, which is unconditional. Upon assuming the Generalized Riemann Hypothesis in Theorem 1.5.2, we exhibit gaps between consecutive zeros of $\zeta_K(s)$ on the critical line which are at least $\sqrt{6} = 2.44949\dots$ times the average spacing. The proof appears in [89].

In 1926, Ingham [48] proved that for $s = 1/2+it$ and $|\alpha|, |\beta| < 1/2$, we have

$$\int_0^T \zeta(s+\alpha)\zeta(1-s+\beta)dt = \int_0^T \left(\zeta(1+\alpha+\beta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \zeta(1-\alpha-\beta)\right)\left(1 + O\big(t^{-\frac{1}{2}+\varepsilon}\big)\right)dt.$$

This 'shifted' moment reveals a beautiful underlying structure which allows one to deduce lower order terms and moments of derivatives of $\zeta(s)$ via differentiation and Cauchy's integral formula. For instance, Ingham's theorem can be used to show that, for fixed $\mu, \nu \in \mathbb{N}$,

$$\int_0^T \zeta^{(\mu)}(\tfrac{1}{2}+it)\zeta^{(\nu)}(\tfrac{1}{2}-it)dt = \frac{(-1)^{\mu+\nu}}{\mu+\nu+1}T(\log T)^{\mu+\nu+1} + O\big(T(\log T)^{\mu+\nu}\big),$$

where $\zeta^{(\mu)}(s)$ denotes the $\mu^{\text{th}}$ derivative of $\zeta(s)$. We make use of a similar shifted moment result for a Dedekind zeta-function of a quadratic number field due to Heap [46] to obtain the mixed second moments of derivatives of $\zeta_K(s)$ on the critical line. We then combine these results with an argument of R.R. Hall [40] to arrive at the conclusion of Theorem 1.5.2.

3.1 Preliminary Results

The following shifted moment result for a Dedekind zeta-function of a quadratic number field has recently been given by Heap [46].

**Theorem 3.1.1.** (Heap) *Let $K$ be the quadratic number field with discriminant $d$. Let $s = 1/2+it$ and $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|, |\beta| \ll 1/\log(\sqrt{|d|}T)$. Then we have*

$$\begin{aligned}
&\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\,dt \qquad\qquad (3.1.1)\\
&= \int_T^{2T} \Bigg\{ \prod_p \left(1-\frac{1}{p^{2+2\alpha+2\beta}}\right) \prod_{p|d}\left(1+\frac{1}{p^{1+\alpha+\beta}}\right)^{-1} \zeta_K^2(1+\alpha+\beta)\\
&\quad + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{6}{\pi^2} \prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1} \prod_{p|d}\left(1-\frac{1}{p^{1+\alpha+\beta}}\right) L^2(1,\chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)\\
&\quad + \frac{1}{d^{\alpha+\beta}}\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{6}{\pi^2} \prod_{p|d}\left(1-\frac{1}{p^2}\right)^{-1} \prod_{p|d}\left(1-\frac{1}{p^{1-\alpha-\beta}}\right) L^2(1,\chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)\\
&\quad + \frac{1}{d^{\alpha+\beta}}\left(\frac{t}{2\pi}\right)^{-2\alpha-2\beta} \prod_p\left(1-\frac{1}{p^{2-2\alpha-2\beta}}\right) \prod_{p|d}\left(1+\frac{1}{p^{1-\alpha-\beta}}\right)^{-1} \zeta_K^2(1-\alpha-\beta)\Bigg\}\,dt\\
&\quad + O\big(|d|^{\varepsilon} C_d T \log T\big)
\end{aligned}$$

*where the constant $C_d$ is defined in* (3.1.2).

*Proof.* This is a consequence of [46, Theorem 1], letting $h = k = 1$. □

The proof of Theorem 1.5.2 requires asymptotic estimates of the mixed second moments of $\zeta_K(\frac{1}{2}+it)$ and $\zeta_K'(\frac{1}{2}+it)$ with a uniform error. We obtain these by way of the following theorem, which is a consequence of Theorem 3.1.1.

**Theorem 3.1.2.** *Let $K$ be the quadratic number field with discriminant $d$. Let $T \geq 2$, and $\mu, \nu$ be non-negative integers. We have*

$$\int_T^{2T} \zeta_K^{(\mu)}(\tfrac{1}{2}+it)\zeta_K^{(\nu)}(\tfrac{1}{2}-it)\,dt$$
$$= \frac{(-1)^{\mu+\nu}(2^{\mu+\nu+1}-1)}{(\mu+\nu+2)(\mu+\nu+1)} 2C_d T(\log T)^{\mu+\nu+2} + O\big(\mu!\nu!|d|^{\varepsilon} C_d T(\log T)^{\mu+\nu+1}\big),$$

*where the constant*

$$C_d := \frac{6}{\pi^2} \prod_{p|d} \left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi_d). \tag{3.1.2}$$

Special cases of Theorem 3.1.2 are known by the work of Motohashi [71] and Weinstein [92], however we require the more general case to prove Theorem 1.5.2.

*Proof of Theorem 3.1.2.* Let $\varepsilon > 0$ be an arbitrary constant, $s = 1/2+it$, and $T \geq 2$ be fixed. We first simplify the integral on the right-hand side of (3.1.1) by considering each factor of each term of the integrand. Since $\alpha+\beta \ll 1/\log(\sqrt{|d|}T)$, it follows that

$$d^{-\alpha-\beta} = 1 + O((\alpha+\beta)|d|^{\varepsilon}).$$

The Euler products on the right-hand side of (3.1.1) can be simplified as

$$\prod_p \left(1-\frac{1}{p^{2\pm(\alpha+\beta)}}\right) = \prod_p \left(1-\frac{1}{p^2}\right)\Big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\Big) = \frac{6}{\pi^2}\Big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\Big),$$

$$\prod_{p|d} \left(1+\frac{1}{p^{1\pm(\alpha+\beta)}}\right)^{-1} = \prod_{p|d} \left(1+\frac{1}{p}\right)^{-1} \Big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\Big),$$

and

$$\prod_{p|d} \left(1-\frac{1}{p^{1\pm(\alpha+\beta)}}\right) = \prod_{p|d} \left(1-\frac{1}{p}\right) \Big(1+O\big((\alpha+\beta)|d|^{\varepsilon}\big)\Big).$$

The factorization given in (1.5.1) implies that

$$\zeta_K\left(1 \pm (\alpha+\beta)\right) = L(1,\chi_d)\zeta\left(1 \pm (\alpha+\beta)\right)\left(1 + O\big((\alpha+\beta)|d|^{\varepsilon}\big)\right)$$

since $|L(\sigma,\chi_d)| \ll \log|d|$ for $|1-\sigma| \ll 1/\log|d|$. (See Section 14 of the book by Davenport [25].) Furthermore, since $t \in [T, 2T]$, we have that

$$\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} = T^{-\alpha-\beta}\left(1 + O(1/\log T)\right).$$

Using these estimates, we find that

$$\begin{aligned}
&\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\,dt \\
&\qquad = \int_T^{2T} \left\{ \frac{6}{\pi^2} \prod_{p|d}\left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi_d)\zeta^2(1+\alpha+\beta) \right\} dt \\
&\qquad\quad + 2\int_T^{2T} \left\{ \frac{6}{\pi^2} \prod_{p|d}\left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi_d)\zeta(1+\alpha+\beta)\zeta(1-\alpha-\beta)T^{-\alpha-\beta} \right\} dt \\
&\qquad\quad + \int_T^{2T} \left\{ \frac{6}{\pi^2} \prod_{p|d}\left(1+\frac{1}{p}\right)^{-1} L^2(1,\chi_d)\zeta^2(1-\alpha-\beta)T^{-2\alpha-2\beta} \right\} dt \\
&\qquad\quad + O\big(|d|^{\varepsilon} C_d T \log T\big) \\
&\qquad := I_1 + 2I_2 + I_3 + O\big(|d|^{\varepsilon} C_d T \log T\big),
\end{aligned}$$

say. Since $\zeta(1-s) = 1/s + O(1)$, we can express the three integrals as

$$I_1 = (\alpha+\beta)^{-2} C_d T + O\big(|d|^{\varepsilon} C_d T \log T\big), \qquad I_2 = -(\alpha+\beta)^{-2} C_d T^{-\alpha-\beta+1} + O\big(|d|^{\varepsilon} C_d T \log T\big),$$

and

$$I_3 = (\alpha+\beta)^{-2} C_d T^{-2\alpha-2\beta+1} + O\big(|d|^{\varepsilon} C_d T \log T\big).$$

Finally, noting that

$$T^{-\delta(\alpha+\beta)} = \sum_{n=0}^{\infty} \frac{(-1)^n \delta^n (\alpha+\beta)^n (\log T)^n}{n!},$$

we simplify $I_1+2I_2+I_3$ to conclude that

$$\int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\, dt = F(\alpha+\beta;T) + O\big(|d|^{\varepsilon} C_d T \log T\big),$$

where

$$F(\alpha+\beta;T) := 2C_d T \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha+\beta)^n (\log T)^{n+2}}{(n+2)!} \{2^{n+1} - 1\}. \tag{3.1.3}$$

We now follow an argument of Ingham [48] to complete the proof. Let

$$R(\alpha,\beta;T) := \int_T^{2T} \zeta_K(s+\alpha)\zeta_K(1-s+\beta)\, dt - F(\alpha+\beta;T). \tag{3.1.4}$$

Then $R(\alpha,\beta;T)$ is an analytic function of the two complex variables $\alpha$ and $\beta$ when $\Re(\alpha), \Re(\beta) < 1/2$, and

$$R(\alpha,\beta;T) = O\big(|d|^{\varepsilon} C_d T \log T\big) \tag{3.1.5}$$

holds by Theorem 3.1.1. Differentiating (3.1.4), it follows that

$$\int_T^{2T} \zeta_K^{(\mu)}(s+\alpha)\zeta_K^{(\nu)}(1-s+\beta)\, dt = \frac{\partial^{\mu+\nu} F(\alpha+\beta;T)}{\partial\alpha^{\mu}\partial\beta^{\nu}} + R_{\mu,\nu}(\alpha,\beta;T), \tag{3.1.6}$$

where $\mu$ and $\nu$ are fixed nonnegative integers and

$$R_{\mu,\nu}(\alpha,\beta;T) := \frac{\partial^{\mu+\nu} R(\alpha,\beta;T)}{\partial\alpha^{\mu}\partial\beta^{\nu}}.$$

Let $\mathscr{C} = \{w \in \mathbb{C}; |w-\alpha| = 1/\log T\}$. By the Cauchy integral formula and (3.1.5), we have

$$\frac{\partial^\mu}{\partial\alpha^\mu} R(\alpha,\beta;T) = \frac{\mu!}{2\pi i}\int_{\mathscr{C}} \frac{R(w,\beta;T)}{(w-\alpha)^{\mu+1}}\,dw$$

$$= O\big(\mu!|d|^\varepsilon C_d T(\log T)^{\mu+1}\big).$$

Appealing to the Cauchy integral formula once more, we deduce that

$$R_{\mu,\nu}(\alpha,\beta;T) := \frac{\partial^{\mu+\nu}}{\partial\alpha^\mu\partial\beta^\nu} R(\alpha,\beta;T) = O\big(\mu!\nu!|d|^\varepsilon C_d T(\log T)^{\mu+\nu+1}\big).$$

Thus (3.1.6), with $\alpha = \beta = 0$, gives

$$\int_0^T \zeta_K^{(\mu)}(\tfrac{1}{2}+it)\zeta_K^{(\nu)}(\tfrac{1}{2}-it)\,dt = \left[\frac{\partial^{\mu+\nu} F(\alpha+\beta;T)}{\partial\alpha^\mu\partial\beta^\nu}\right]_{\alpha=\beta=0} + O\big(\mu!\nu!|d|^\varepsilon C_d T(\log T)^{\mu+\nu+1}\big), \tag{3.1.7}$$

and it remains only to calculate the first term on the right-hand side. By differentiating (3.1.3) with respect to $\alpha$ and $\beta$ and simplifying, we determine that

$$\left[\frac{\partial^{\mu+\nu} F(\alpha+\beta;T)}{\partial\alpha^\mu\partial\beta^\nu}\right]_{\alpha=\beta=0} = \frac{(-1)^{\mu+\nu}(2^{\mu+\nu+1}-1)}{(\mu+\nu+2)(\mu+\nu+1)} 2C_d T(\log T)^{\mu+\nu+2}. \tag{3.1.8}$$

Theorem 3.1.2 now follows upon inserting (3.1.8) into (3.1.7). $\square$

We now demonstrate how to obtain the lower bound in Theorem 1.5.2. The proof is a variation of a method of R. R. Hall [40] using some ideas of Bredberg [6]. We begin by defining the function

$$f(t) := e^{ivt\log T}\zeta_K(\tfrac{1}{2}+it), \tag{3.1.9}$$

where $v$ is a real constant that will be chosen later. By Stirling's formula, $f(t)$ mimics the analogue of the Hardy $Z$-function for $\zeta_K(s)$. Fix $K$, and let $\widetilde{\gamma}$ denote an ordinate of a zero of $\zeta_K(s)$ on the critical line $\Re(s) = 1/2$. Note that $f(t)$ has the same zeros as $\zeta_K(\frac{1}{2}+it)$, that is, $f(t) = 0$ if and only if $t = \widetilde{\gamma}$. Let $\{\widetilde{\gamma}_1, \widetilde{\gamma}_2, \ldots, \widetilde{\gamma}_N\}$ denote the set of distinct zeros of $f(t)$ in the interval $[T, 2T]$ arranged in non-decreasing order and ignoring multiplicity. Furthermore, let

$$\kappa_T = \max\{\tilde{\gamma}_{n+1} - \tilde{\gamma}_n \,:\, T+1 \le \tilde{\gamma}_n \le 2T-1\},$$

and note that $\lambda_K \ge \limsup_{T\to\infty} \kappa_T$. Without loss of generality, we may assume that

$$\tilde{\gamma}_1 - T \ll 1 \quad \text{and} \quad 2T - \tilde{\gamma}_N \ll 1, \tag{3.1.10}$$

as otherwise there exist zeros $\widetilde{\gamma}_0 \le \widetilde{\gamma}_1$ and $\widetilde{\gamma}_{N+1} \ge \widetilde{\gamma}_N$ such that $\widetilde{\gamma}_0 - \widetilde{\gamma}_1$ and $\widetilde{\gamma}_{N+1} - \widetilde{\gamma}_N$ are $\gg 1$, and Theorem 1.5.2 holds for this reason. In order to obtain a lower bound on $\kappa_T$, we require the following lemma.

**Lemma 3.1.3.** *Let $y : [a, b] \to \mathbb{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then*

$$\int_a^b |y(x)|^2 \, dx \le \left(\frac{b-a}{\pi}\right)^2 \int_a^b |y'(x)|^2 \, dx.$$

*Proof.* This is a variation of a well-known inequality of Wirtinger [43, Theorem 256] due to Bredberg [6, Corollary 1]. □

## 3.2 The Proof of Theorem 1.5.2

With the above setup, we now prove Theorem 1.5.2.

*Proof of Theorem 1.5.2.* Let $\varepsilon > 0$ be a small positive constant which may vary from line to line, and let $f(t)$ be the function defined in (3.1.9). By the definition of $\kappa_T$, for each pair of consecutive zeros of $f(t)$ in the interval $[T, 2T]$, we have

$$\int_{\widetilde{\gamma}_n}^{\widetilde{\gamma}_{n+1}} |f(t)|^2 \, dt \le \frac{\kappa_T^2}{\pi^2} \int_{\widetilde{\gamma}_n}^{\widetilde{\gamma}_{n+1}} |f'(t)|^2 \, dt. \tag{3.2.1}$$

Summing both sides of the equation in (3.2.1) over $n$ for $n = 1, 2, \ldots, N-1$, it follows that

$$\int_{\widetilde{\gamma}_1}^{\widetilde{\gamma}_N} |f(t)|^2 \, dt \le \frac{\kappa_T^2}{\pi^2} \int_{\widetilde{\gamma}_1}^{\widetilde{\gamma}_N} |f'(t)|^2 \, dt.$$

By Weyl's bound for the zeta-function,

$$\zeta(\tfrac{1}{2}+it) \ll t^{\frac{1}{6}+\varepsilon},$$

and the subconvexity bound

$$L(\tfrac{1}{2}+it, \chi_d) \ll |td|^{\frac{3}{16}+\varepsilon}$$

due to Heath-Brown [47], we see that, for $T \le t \le 2T$ and $\varepsilon > 0$,

$$|f(t)| \ll t^{\frac{17}{48}+\varepsilon} |d|^{\frac{3}{16}+\varepsilon}.$$

Therefore, by the assumption in (3.1.10), we have

$$\int_T^{2T} |f(t)|^2 \, dt \le \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |f'(t)|^2 \, dt + O(|d|^{\frac{3}{8}+\varepsilon} T^{\frac{17}{24}+\varepsilon}). \tag{3.2.2}$$

Note that $|f(t)|^2 = |\zeta_K(\frac{1}{2}+it)|^2$ and

$$|f'(t)|^2 dt = |\zeta_K'(\tfrac{1}{2}+it)|^2 + v^2 \log^2 T |\zeta_K(\tfrac{1}{2}+it)|^2 + 2v \log T \cdot \mathrm{Re}\Big(\zeta_K'(\tfrac{1}{2}+it)\overline{\zeta_K(\tfrac{1}{2}+it)}\Big). \tag{3.2.3}$$

Theorem 3.1.2 implies that

$$\int_T^{2T} |\zeta_K(\tfrac{1}{2}+it)|^2 dt = C_d T \log^2 T + O\big(|d|^\varepsilon C_d T \log T\big), \tag{3.2.4}$$

$$\int_T^{2T} \zeta_K'(\tfrac{1}{2}+it)\overline{\zeta_K(\tfrac{1}{2}+it)} dt = -C_d T \log^3 T + O\big(|d|^\varepsilon C_d T \log^2 T\big), \tag{3.2.5}$$

and

$$\int_T^{2T} |\zeta_K'(\tfrac{1}{2}+it)|^2 dt = \frac{7}{6} C_d T \log^4 T + O\big(|d|^\varepsilon C_d T \log^3 T\big), \tag{3.2.6}$$

where $C_d$ is the constant in (3.1.2). By combining the estimates in (3.2.2) – (3.2.6), we find that

$$\frac{\kappa_T^2}{\pi^2} \geq \frac{6}{6v^2 - 12v + 7} \frac{1}{\log^2 T} \bigg(1 + O\big(|d|^\varepsilon \log^{-1} T\big)\bigg),$$

uniformly for $|d| \leq T^{\frac{7}{9}-\varepsilon}$. The choice of $v = 1$ minimizes $6v^2 - 12v + 7$, the minimum value being 1. We conclude that

$$\kappa_T \geq \frac{\sqrt{6}\pi}{\log(\sqrt{|d|}T)} \bigg(1 + O\big(|d|^\varepsilon \log^{-1} T\big)\bigg).$$

This completes the proof. $\square$

# 4 THE PROOF OF THEOREM 1.5.4

In this chapter, we prove Theorem 1.5.4, that is, under the assumption of some standard conjectures, we show that

$$\int_0^T |L(\tfrac{1}{2}+it,\pi_1)|^{2k_1}\cdots|L(\tfrac{1}{2}+it,\pi_r)|^{2k_r}dt \ll T(\log T)^{k_1^2+\cdots+k_r^2+\varepsilon}$$

for any $k_1,\ldots,k_r>0$ and every $\varepsilon>0$ when $T$ is sufficiently large. This result was proved in collaboration with Micah B. Milinovich and appears in [63].

There are a couple of aspects which make the proof of Theorem 1.5.4 different than the proof of the analogous result for the Riemann zeta-function. First of all, we need to understand the correlations of the coefficients of distinct $L$-functions averaged over the primes. Secondly, we need to handle the contribution of these coefficients at the prime powers. In [85], assuming the Riemann hypothesis, an inequality for the real part of the logarithm of the Riemann zeta-function is derived which depends only on the primes. In the case of $\zeta(s)$, one can handle the contribution of the primes powers relatively easily. If we were willing to assume the Ramanujan-Petersson Conjecture (given below) and the Generalized Riemann Hypothesis for the symmetric square $L$-functions, then we could similarly derive an inequality involving only the primes for the real part of the logarithms of the $L$-functions in Theorem 1.5.4. In order to circumvent these additional assumptions, we must estimate the contribution from the prime powers in a different way. To this end, we use a partial result toward the Ramanujan-Petersson Conjecture for automorphic $L$-functions due to Luo, Rudnick, and Sarnak [61] and also Hypothesis H (given below) which is known to hold for automorphic $L$-functions of small degree. Ideas similar to these were used for degree two $L$-functions in [64].

## 4.1 Hypotheses and Conjectures

In this section we collect some hypotheses and conjectures. First, recall from Subsection 1.3.1 that for $\Re(s) > 1$, we have

$$L(s,\pi) := \sum_{n=1}^{\infty} \frac{a_\pi(n)}{n^s} = \prod_{\substack{p \\ p \text{ prime}}} \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}$$

and

$$-\frac{L'}{L}(s,\pi) = \sum_{p^\ell,\, \ell \geq 1} \frac{\left(\alpha_1^\ell(p) + \cdots + \alpha_m^\ell(p)\right) \log p}{p^{\ell s}} =: \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s}.$$

**Ramanujan-Petersson Conjecture.** The local parameters $\alpha_j(p)$ satisfy $|\alpha_j(p)| = 1$ for all but a finite number of primes $p$.

In general, this conjecture is open. Towards the Ramanujan-Petersson Conjecture, Luo, Rudnick, and Sarnak [61] have shown that

$$|\alpha_j(p)| \leq p^{1/2 - 1/(m^2+1)}$$

for all $p$. It follows that

$$|\Lambda_\pi(n)| < m\,\Lambda(n)\, n^{1/2 - 1/(m^2+1)} \tag{4.1.1}$$

where $\Lambda(n)$ is the Von Mandgoldt function, defined by $\Lambda(n) = \log p$ if $n = p^j, j \geq 1$, and $\Lambda(n) = 0$ otherwise. The bound in (4.1.1) is crucial to the proofs of Theorem 1.5.4 and Theorem 1.6.1.

We will make use of Hypothesis H of Rudnick and Sarnak [79].

**Hypothesis H.** Let $j \geq 2$ be fixed, and let $\pi$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}(m)$ over $\mathbb{Q}$. Then we have

$$\sum_{p} \frac{|\Lambda_\pi(p^j)|^2}{p^j} < \infty.$$

Hypothesis H is known to hold for automorphic $L$-functions of small degree.

**Theorem 4.1.1.** *Hypothesis H is true for $m \leq 4$.*

*Proof.* The case $m = 1$ is trivial, the case $m = 2$ follows from the work of Kim and Sarnak [56], the case $m = 3$ is due to Rudnick and Sarnak [79], and the case $m = 4$ is due to Kim [55]. □

Given distinct automorphic $L$-functions $L(s, \pi)$ and $L(s, \pi')$, we need to understand the correlation of their Dirichlet series coefficients averaged over the primes. Selberg [82] has made the following conjecture (in a different context).

**Selberg's Orthogonality Conjectures.** Let $\pi$ and $\pi'$ be two irreducible unitary cuspidal automorphic representations of $\mathrm{GL}(m)$ and $\mathrm{GL}(m')$ over $\mathbb{Q}$, respectively, and let $x \geq 3$. Then

$$\sum_{p \leq x} \frac{a_\pi(p)\overline{a_{\pi'}(p)}}{p} = \sum_{p \leq x} \frac{\Lambda_\pi(p)\overline{\Lambda_{\pi'}(p)}}{p \log^2 p} = \begin{cases} \log\log x + O(1), & \text{if } \pi \cong \pi', \\ O(1), & \text{if } \pi \not\cong \pi'. \end{cases}$$

The following result allows us to use Selberg's Orthogonality Conjectures in the proofs of Theorems 1.5.4 and 1.6.1.

**Theorem 4.1.2.** *Let $\pi$ and $\pi'$ be two irreducible unitary cuspidal automorphic representations of $\mathrm{GL}(m)$ and $\mathrm{GL}(m')$ over $\mathbb{Q}$, respectively. If $L(s, \pi)$ and $L(s, \pi')$ satisfy Hypothesis H, then the coefficients of these $L$-functions satisfy Selberg's orthogonality conjectures. In particular, Selberg's orthogonality conjectures hold if $\max(m, m') \leq 4$.*

*Proof.* This was proved in the special case where at least one of $\pi$ or $\pi'$ is self-contragredient in [58, 59], and in full generality by Liu and Ye in [60]. See also Avdispahić and Smajlović [3]. $\square$

## 4.2 Lemmas

In this section, we state three lemmas that will be used in the proof of Theorem 1.5.4.

**Lemma 4.2.1.** *If $\{b_n\}$ is a sequence of complex numbers such that $\sum |b_n|$ and $\sum n|b_n|^2$ are convergent, then*

$$\int_0^T \left| \sum_{n=1}^{\infty} b_n n^{-it} \right|^2 dt = T \sum_{n=1}^{\infty} |b_n|^2 + O\Big( \sum_{n=1}^{\infty} n|b_n|^2 \Big)$$

*where the implied constant is absolute.*

*Proof.* This is Montgomery and Vaughan's mean-value theorem for Dirichlet polynomials (see Corollary 3 of [67]). $\square$

**Lemma 4.2.2.** *Let $T$ be large, $x \geq 2$, and let $\ell$ and $j$ be natural numbers satisfying $x^\ell \leq T^j$. Then for any complex numbers $b(p)$ we have*

$$\frac{1}{T} \int_T^{2T} \left| \sum_{p^j \leq x} \frac{b(p)}{p^{j(\sigma+it)}} \right|^{2\ell} dt \ll \ell! \left\{ \sum_{p^j \leq x} \frac{|b(p)|^2}{p^{2j\sigma}} \right\}^\ell$$

*where $j$ is fixed and the sum runs over the primes $p$.*

*Proof.* This is a consequence of Lemma 4.2.1. The case $j = 1$ essentially corresponds to Lemma 3 of Soundararajan [85]. For any $s \in \mathbb{C}$, write

$$\left\{ \sum_{p \leq y} \frac{b(p)}{p^s} \right\}^\ell = \sum_{n \leq y^\ell} \frac{\beta_{y,\ell}(n)}{n^s},$$

where $\beta_{y,\ell}(n) = 0$ unless $n$ is the product of $\ell$ (not necessarily distinct) primes, all less than or equal to $y$. In the later case, writing

$$n = \prod_{i=1}^{r} p_i^{\alpha_i},$$

where $p_1 < \ldots < p_r \leq y$ and $\alpha_1 + \cdots + \alpha_r = \ell$, we have

$$\beta_{y,\ell}(n) = \binom{\ell}{\alpha_1, \ldots, \alpha_r} \prod_{i=1}^{r} b(p_i)^{\alpha_i}.$$

Thus, we have

$$\int_T^{2T} \left| \sum_{p \leq y} \frac{b(p)}{p^{j(\sigma+it)}} \right|^{2\ell} dt = \int_T^{2T} \left| \sum_{n \leq y^\ell} \frac{\beta_{y,\ell}(n)}{n^{j\sigma+jit}} \right|^2 dt = \frac{1}{j} \int_{jT}^{2jT} \left| \sum_{n \leq y^\ell} \frac{\beta_{y,\ell}(n)}{n^{j\sigma+iu}} \right|^2 du,$$

where in the last step we have made the variable change $u = jt$. If $y^\ell \leq T$, then Lemma 4.2.1 implies that

$$\int_T^{2T} \left| \sum_{p \leq y} \frac{b(p)}{p^{j(\sigma+it)}} \right|^{2\ell} dt \ll \frac{2jT - jT}{j} \sum_{n \leq y^\ell} \frac{|\beta_{y,\ell}(n)|^2}{n^{2j\sigma}} \ll T \sum_{n \leq y^\ell} \frac{|\beta_{y,\ell}(n)|^2}{n^{2j\sigma}}.$$

We now follow the combinatorial argument appearing in the proof of Lemma 3 of [85]. We have

$$\begin{aligned}
\sum_{n \leq y^\ell} \frac{|\beta_{y,\ell}(n)|^2}{n^{2j\sigma}} &= \sum_{p_1 < \ldots < p_r \leq y} \sum_{\substack{\alpha_1, \ldots, \alpha_r \geq 1 \\ \alpha_1 + \ldots + \alpha_r = \ell}} \binom{\ell}{\alpha_1, \ldots, \alpha_r}^2 \frac{|b(p_1)|^{2\alpha_1} \cdots |b(p_r)|^{2\alpha_r}}{p_1^{2j\sigma\alpha_1} \cdots p_r^{2j\sigma\alpha_r}} \\
&\leq \ell! \sum_{p_1 < \ldots < p_r \leq y} \sum_{\substack{\alpha_1, \ldots, \alpha_r \geq 1 \\ \alpha_1 + \ldots + \alpha_r = \ell}} \binom{\ell}{\alpha_1, \ldots, \alpha_r} \frac{|b(p_1)|^{2\alpha_1} \cdots |b(p_r)|^{2\alpha_r}}{p_1^{2j\sigma\alpha_1} \cdots p_r^{2j\sigma\alpha_r}} \\
&= \ell! \left( \sum_{p \leq y} \frac{|b(p)|^2}{p^{2j\sigma}} \right)^\ell,
\end{aligned}$$

that is

$$\sum_{n\le y^\ell} \frac{|\beta_{y,\ell}(n)|^2}{n^{2j\sigma}} \ll \ell! \left\{ \sum_{p\le y} \frac{|b(p)|^2}{p^{2j\sigma}} \right\}^\ell .$$

Combining estimates, the lemma follows. □

**Lemma 4.2.3.** *Assume that either $L(s,\pi)$ is the Riemann zeta-function or that $\Phi(s,\pi)$ has no pole or zero at $s=0,1$. Let $\lambda_0 = 0.4912\ldots$ denote the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$. Then, assuming the Generalized Riemann Hypothesis for $L(s,\pi)$, for all $\lambda_0 \le \lambda \le \log x/2$ and $\log x \ge 2$, we have*

$$\log |L(\tfrac{1}{2}+it,\pi)| \le \Re \sum_{n\le x} \frac{\Lambda_\pi(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log n} \frac{\log x/n}{\log x} + \frac{(1+\lambda)}{2}\frac{m\log T}{\log x} + O\Big(\frac{1}{\log x}\Big)$$

*for $T \le t \le 2T$ and $T$ sufficiently large, where the implied constant in the error term depends only on $\pi$.*

*Proof.* The case where $L(s,\pi)$ corresponds to the Riemann zeta-function is due to Soundararajan [85], and the other cases are a consequence of Theorem 2.1 of Chandee [12]. □

## 4.3 The Frequency of Large Values of $\prod_{1\le i\le k} |L(\frac{1}{2}+it,\pi_i)|$

In this section, we state and prove a value distribution result for a linear combination of distinct primitive $L$-functions which will be used to deduce Theorem 1.5.4. This result is an analogue of the main theorem in [85], and the proof given here is an expanded version of the argument appearing in [63].

Let $L(s,\pi_1),\ldots,L(s,\pi_r)$ be $r$ distinct primitive $L$-functions (as in Theorem 1.5.4) of degrees $m_1,\ldots,m_r$, respectively, let

$$\Delta = \max\left\{m_1^2+1,\ldots,m_r^2+1\right\},$$

and let

$$B = k_1 m_1 + \cdots + k_r m_r + 1. \tag{4.3.1}$$

Define the set

$$\mathcal{A}(T,V) = \{t \in [T,2T] : k_1 \log|L(\tfrac{1}{2}+it,\pi_1)| + \cdots + k_r \log|L(\tfrac{1}{2}+it,\pi_r)| \geq V\}$$

and the quantity

$$W = (k_1^2 + \cdots + k_r^2) \log\log T.$$

Note that

$$\begin{aligned} \int_T^{2T} |L(\tfrac{1}{2}+it,\pi_1)|^{2k_1} \cdots |L(\tfrac{1}{2}+it,\pi_r)|^{2k_r}\,dt &= -\int_{-\infty}^{\infty} \exp(2V)\,d\,\mathrm{meas}(\mathcal{A}(T,V)) \\ &= 2\int_{-\infty}^{\infty} \exp(2V)\,\mathrm{meas}(\mathcal{A}(T,V))\,dV. \end{aligned} \tag{4.3.2}$$

To prove Theorem 1.5.4, it suffices to estimate the measure of $\mathcal{A}(T,V)$ for all $V \geq 3$ when $T$ is large. Note that the definitions of $\mathcal{A}(T,V)$ and $W$ depend on our choices of $k_1,\ldots,k_r$, which we consider to be fixed throughout the proof Proposition 4.3.1 below.

We prove estimates for the size of $\mathcal{A}(T,V)$ using Lemmas 4.2.2 and 4.2.3. The contribution to the size of $\mathcal{A}(T,V)$ coming from the primes in the sum on the right-hand side of the inequality in Lemma 4.2.3 is estimated following the method of Soundararajan in [85], and the contribution from the prime powers $p^j$ with $j > \Delta$ is estimated trivially. More care is necessary to handle the contribution from the prime powers $p^j$ with $2 \leq j \leq \Delta$, and this is where we appeal to (4.1.1) and Hypothesis H. This allows us to circumvent using the Ramanujan-Petersson Conjecture.

As might be expected, the proof of Theorem 1.5.4 relies on understanding the correlations between coefficients of distinct automorphic $L$-functions. The key ingredient to the proof of the proposition below (and hence Theorem 1.5.4) is the fact that the Selberg orthogonality conjectures imply that

$$\sum_{p\le z} \frac{|k_1\Lambda_{\pi_1}(p)+\cdots+k_r\Lambda_{\pi_r}(p)|^2}{p\log^2 p} = (k_1^2+\cdots+k_r^2)\log\log z + O(1) \tag{4.3.3}$$

as $z\to\infty$, which can be seen by expanding the square on the left hand side of (4.3.3).

**Proposition 4.3.1.** *Let $L(s,\pi_1),\ldots,L(s,\pi_r)$ be L-functions attached to distinct irreducible cuspidal automorphic representations, $\pi_j$, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ with unitary central character, and assume that these L-functions satisfy the Generalized Riemann Hypothesis. If $\max_{1\le j\le r} m_j \le 4$ or each of the L-functions satisfies Hypothesis H, then the following inequalities hold. If $\sqrt{W}\le V\le \frac{W}{B^2}$, we have*

$$\mathrm{meas}(\mathcal{A}(T,V)) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V^2}{W}\left(1-\frac{4}{\log W}\right)\right);$$

*if $\frac{W}{B^2}\le V\le \frac{1}{2B^2}W\log W$, we have*

$$\mathrm{meas}(\mathcal{A}(T,V)) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V^2}{W}\left(1-\frac{7B^2V}{4W\log W}\right)^2\right);$$

*and if $\frac{1}{2B^2}W\log W\le V$ , we have*

$$\mathrm{meas}(\mathcal{A}(T,V)) \ll T\exp\left(-\frac{1}{129B^2}V\log V\right)$$

*for any $k_1,\ldots,k_r>0$ when $T$ is sufficiently large.*

*Proof of Proposition 4.3.1.* Our proof is similar to the proof of the main theorem of Soundararajan in [85], and our notation follows that of [85] and Chandee [13]. Let $L(s,\pi)$ be a primitive $L$-function of degree $m$. Let $\lambda=\lambda_0<1/2$ and $\varepsilon<(1-2\lambda_0)/3$. Choosing $x=(\log T)^{1-\varepsilon}$, it follows from Lemma 4.2.3 and (4.1.1) that

$$\begin{aligned}\log|L(\tfrac{1}{2}+it,\pi)| &\le m(\log T)^{1-\varepsilon}+\frac{(1+\lambda_0)m\log T}{2(1-\varepsilon)\log\log T}+O\left(\frac{1}{(1-\varepsilon)\log\log T}\right)\\ &\le \frac{3m}{4}\frac{\log T}{\log\log T}\end{aligned}$$

for sufficiently large $T$. Therefore, we see that

$$k_1\log|L(\tfrac{1}{2}+it,\pi_1)|+\cdots+k_r\log|L(\tfrac{1}{2}+it,\pi_r)|\le\frac{3(k_1m_1+\cdots+k_rm_r)}{4}\frac{\log T}{\log\log T}$$

when $T$ is large. Recalling the definition of $B$ in (4.3.1), we may assume that

$$\sqrt{W}\le V\le\frac{3(B-1)}{4}\frac{\log T}{\log\log T}$$

while proving the proposition. Note that $B>1$ (a fact that is useful when deriving the estimates below). Define a parameter $A$ as

$$A=\begin{cases}\dfrac{B}{2}\log W, & \text{if } \sqrt{W}\le V\le\frac{W}{B^2},\\[2ex] \dfrac{1}{2BV}W\log W, & \text{if } \frac{W}{B^2}<V\le\frac{1}{2B^2}W\log W,\\[2ex] B, & \text{if } V>\frac{1}{2B^2}W\log W,\end{cases}$$

and let $x=T^{A/V}$ and $z=x^{1/\log\log T}$. Choosing $\lambda=1/2$ in Lemma 4.2.3, we deduce that

$$\begin{aligned}&k_1\log|L(\tfrac{1}{2}+it,\pi_1)|+\cdots+k_r\log|L(\tfrac{1}{2}+it,\pi_r)|\\ &\qquad\le|S_1(t)|+|S_1^\star(t)|+\sum_{2\le j\le\Delta}|S_j(t)|+\frac{3(B-1)}{4}\frac{V}{A}+O(1),\end{aligned}\tag{4.3.4}$$

where

$$S_1(t) = \sum_{p \le z} \frac{(k_1\Lambda_{\pi_1}(p) + \cdots + k_r\Lambda_{\pi_r}(p))}{p^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log p} \frac{\log(x/p)}{\log x},$$

$$S_1^\star(t) = \sum_{z < p \le x} \frac{(k_1\Lambda_{\pi_1}(p) + \cdots + k_r\Lambda_{\pi_r}(p))}{p^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log p} \frac{\log(x/p)}{\log x},$$

and

$$S_j(t) = \sum_{p^j \le x} \frac{(k_1\Lambda_{\pi_1}(p^j) + \cdots + k_r\Lambda_{\pi_r}(p^j))}{p^{j(\frac{1}{2}+\frac{\lambda}{\log x}+it)}\log p^j} \frac{\log(x/p^j)}{\log x}$$

for $2 \le j \le \Delta$. The contribution from the prime powers for which $j > \Delta$ is $O(1)$. Indeed, (4.1.1) implies that

$$|k_1\Lambda_{\pi_1}(p^j) + \cdots + k_r\Lambda_{\pi_r}(p^j)| \ll (k_1 m_1 + \cdots + k_r m_r)(\log p)p^{j/2-j/\Delta},$$

and hence

$$\begin{aligned}
\sum_{\substack{p^j \le x \\ j > \Delta}} \frac{|k_1\Lambda_{\pi_1}(p^j) + \cdots + k_r\Lambda_{\pi_r}(p^j)|}{p^{j(\frac{1}{2}+\frac{\lambda}{\log x}+it)}\log p^j} \frac{\log(x/p^j)}{\log x} &\ll \sum_{\substack{p^j \le x \\ j > \Delta}} \frac{|k_1\Lambda_{\pi_1}(p^j) + \cdots + k_r\Lambda_{\pi_r}(p^j)|}{jp^{j/2}\log p} \\
&\ll (B-1)\sum_{\substack{p^j \le x \\ j > \Delta}} \frac{1}{p^{j/\Delta}} \\
&\ll B.
\end{aligned}$$

Let

$$V_1 := V\left(1 - \frac{7(B-1)}{8A}\right), \quad V_1^\star = V_j := \frac{(B-1)V}{8\Delta A}$$

for $2 \le j \le \Delta$. Note that if $t \in \mathcal{A}(T, V)$, then at least one of the following inequalities holds:

$$|S_1^\star(t)| \ge V_1^\star \quad \text{or} \quad |S_j(t)| \ge V_j$$

for some $j = 1, 2, \ldots, \Delta$. To see why, suppose (towards contradiction) that $|S_1^\star(t)| < V_1^\star$ and $|S_j(t)| < V_j$ for all $1 \leq j \leq \Delta$. Then, if $t \in [T, 2T]$, we have

$$\begin{aligned}
V &\leq k_1 \log|L(\tfrac{1}{2}+it, \pi_1)| + \cdots + k_r \log|L(\tfrac{1}{2}+it, \pi_r)| \\
&\leq |S_1(t)| + |S_1^\star(t)| + \sum_{2 \leq j \leq \Delta} |S_j(t)| + \frac{3(B-1)}{4}\frac{V}{A} + O(1) \\
&< V\left(1 - \frac{7(B-1)}{8A}\right) + \frac{(B-1)V}{8A} + \frac{3(B-1)}{4}\frac{V}{A} + O(1) \\
&= V + O(1),
\end{aligned}$$

a contradiction. If we define

$$N_j(T, V_j) := \operatorname{meas}\{t \in [T, 2T] : |S_j(t)| \geq V_j\}$$

for $j = 1, 2, \ldots, \Delta$ and define $N_1^\star(T, V_1^\star)$ similarly, then we can bound $N_j(T, V_j)$ and $N_1^\star(T, V_1^\star)$ using Lemma 4.2.2 since Chebyshev's inequality implies that

$$N_j(T, V_j) \leq (V_j)^{-2\ell} \int_T^{2T} |S_j(t)|^{2\ell}\, dt$$

and

$$N_1^\star(T, V_1^\star) \leq (V_1^\star)^{-2\ell} \int_T^{2T} |S_1^\star(t)|^{2\ell}\, dt$$

for every non-negative integer $\ell$.

Let us first estimate $N_1(T,V_1)$. Letting $\ell$ be any natural number such that $z^\ell \le T$, Lemma 4.2.2 and (4.3.3) imply that

$$\begin{aligned}
\int_T^{2T} |S_1(t)|^{2\ell} dt &\ll T\ell! \left( \sum_{p\le z} \frac{|k_1\Lambda_{\pi_1}(p) + \cdots + k_r\Lambda_{\pi_r}(p)|^2}{p\log^2 p} \right)^\ell \\
&\ll T\ell! \left( (k_1^2 + \cdots + k_r^2)\log\log z + O(1) \right)^\ell \\
&\ll T\ell! \left( (k_1^2 + \cdots + k_r^2)\log\log T \right)^\ell \\
&\ll T\sqrt{\ell} \left( \frac{\ell(k_1^2 + \cdots + k_r^2)\log\log T}{e} \right)^\ell \\
&\ll T\sqrt{\ell} \left( \frac{\ell W}{e} \right)^\ell .
\end{aligned}$$

Thus we have

$$N_1(T,V_1) \ll T\sqrt{\ell} \left( \frac{\ell W}{eV_1^2} \right)^\ell . \tag{4.3.5}$$

We consider separately the two cases where $V \le \frac{W^2}{B^4}$ and $V > \frac{W^2}{B^4}$. In the first case, we choose $\ell = \lfloor \frac{V_1^2}{W} \rfloor$ in (4.3.5). This choice of $\ell$ is permissible since $\ell = \lfloor \frac{V_1^2}{W} \rfloor \le \frac{V\log\log T}{A}$ for each range of $V$, which we now justify.

**Case 1:** $\sqrt{W} \le V \le \frac{W}{B^2}$

In this range, $A = \frac{B}{2}\log W$, and we have

$$\ell \le \frac{V_1^2}{W} \le \frac{V^2}{W} \le V \le \frac{V\log\log T}{A}$$

when $T$ is sufficiently large. Thus $\ell = \lfloor \frac{V_1^2}{W} \rfloor$ is permissible in the range $\sqrt{W} \le V \le \frac{W}{B^2}$.

**Case 2:** $\frac{W}{B^2} < V \le \frac{1}{2B^2} W\log W$

In this range, $A = \frac{1}{2BV} W\log W$. For large $T$, we have

$$\log W \le 2B\log\log T.$$

Thus, it follows that

$$\frac{V^2}{W} \leq \frac{V^2}{W} \cdot \frac{2B \log\log T}{\log W} = \frac{V \log\log T}{A}.$$

Thus $\ell = \lfloor \frac{V_1^2}{W} \rfloor$ is permissible in the range $\frac{W}{B^2} < V \leq \frac{1}{2B^2} W \log W$.

**Case 3:** $\frac{1}{2B^2} W \log W < V \leq \frac{W^2}{B^4}$

In this range, $A = B$. Recalling the definition of $B$ given in (4.3.1), we see that

$$\frac{W}{B^3} \leq \frac{W}{k_1^2 + \cdots + k_r^2} = \log\log T.$$

It follows that

$$\frac{V^2}{W} \leq \frac{VW}{B^4} \leq \frac{V \log\log T}{B}.$$

Therefore $\ell = \lfloor \frac{V_1^2}{W} \rfloor$ is permissible in the range $\frac{1}{2B^2} W \log W < V \leq \frac{W^2}{B^4}$. Thus we may take $\ell = \lfloor \frac{V_1^2}{W} \rfloor$ in (4.3.5) and find

$$N_1(T, V_1) \ll T \frac{V}{\sqrt{W}} \exp\left(-\frac{V_1^2}{W}\right), \qquad V \leq \frac{W^2}{B^4}. \tag{4.3.6}$$

When $V > \frac{W^2}{B^4}$, we choose $\ell = \lfloor 10V \rfloor$ in (4.3.5). This choice of $\ell$ is permissible, since for sufficiently large $T$

$$10V \leq \frac{V \log\log T}{B},$$

and $B = A$ in this range. Thus, taking $\ell = \lfloor 10V \rfloor$ in (4.3.5), we have

$$\begin{aligned}
N_1(T, V_1) &\ll T\sqrt{10V}\left(\frac{10VW}{eV_1^2}\right)^{10V} \\
&\ll T\exp\left(\frac{1}{2}\log V - 10V\log\left(\frac{eV_1^2}{10VW}\right)\right) \\
&\ll T\exp\left(\frac{1}{2}\log V - 10V\log\left(\frac{eV(1-\frac{7(B-1)}{8B})^2}{10W}\right)\right) \\
&\ll T\exp\left(\frac{1}{2}\log V - 10V\log V + 10V\log W\right).
\end{aligned}$$

Since $V > \frac{W^2}{B^4}$, we have $2\log W < \log V + 4\log B$ . Thus

$$10\log W < 5\log V + 20\log B.$$

Hence

$$\begin{aligned}
N_1(T, V_1) &\ll T\exp\left(\frac{1}{2}\log V - 10V\log V + 10V\log W\right) \\
&\ll T\exp\left(V\log V - 10V\log V + 5V\log V\right)
\end{aligned}$$

that is,

$$N_1(T, V_1) \ll T\exp(-4V\log V), \qquad V > \frac{W^2}{B^4}. \tag{4.3.7}$$

Combining (4.3.6) and (4.3.7), we conclude that

$$N_1(T, V_1) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V_1^2}{W}\right) + T\exp(-4V\log V) \tag{4.3.8}$$

for all $V$.

Next, we find an upper bound for $N_1^\star(T, V_1^\star)$. For any natural number $\ell$ with $x^\ell \le T$, Lemma 4.2.2 and (4.3.3) imply that

$$\begin{aligned}\int_T^{2T} |S_1^\star(t)|^{2\ell}\, dt &\ll T\ell! \left( \sum_{z \le p \le x} \frac{|k_1\Lambda_{\pi_1}(p) + \cdots + k_r\Lambda_{\pi_r}(p)|^2}{p \log^2 p} \right)^\ell \\ &\ll T\ell! \big( (k_1^2 + \cdots + k_r^2)(\log\log x - \log\log z) + O(1) \big)^\ell \\ &\ll T\big( \ell(k_1^2 + \cdots + k_r^2) \log\log\log T + O(1) \big)^\ell \\ &\ll T\big( 2\ell(k_1^2 + \cdots + k_r^2) \log\log\log T \big)^\ell\end{aligned}$$

when $T$ is large. Choosing $\ell = \lfloor \frac{V}{A} \rfloor$, which is certainly less than or equal to $\frac{V}{A}$, we find that

$$\begin{aligned}N_1^\star(T, V_1^\star) &\ll TV^{-2\ell}(8\Delta A)^{2\ell}(2\ell \log\log\log T)^\ell \\ &\ll TV^{-2V/A}A^{2V/A} \left( \frac{2V}{A} \log\log\log T \right)^{V/A} \\ &\ll TV^{-2V/A}V^{V/A}A^{2V/A}A^{-V/A}(\log\log\log T)^{V/A} \\ &= TV^{-V/A}A^{V/A}(\log\log\log T)^{V/A}.\end{aligned}$$

Since $\sqrt{W} \le V$, we have $W = (k_1^2 + \cdots + k_r^2)\log\log T \le V^2$. Thus

$$\log\log\log T \le \log V^2 - \log(k_1^2 + \cdots + k_r^2) \ll \log V$$

and

$$A \le \frac{B \log W}{2} \le \frac{B \log V^2}{2} \ll \log V.$$

Thus $A \log\log\log T \ll \log^2 V$. Furthermore, if $T$ is sufficiently large, then $\log^2 V \leq V^{1/2}$. Thus

$$\begin{aligned} N_1^{\star}(T, V_1^{\star}) &\ll TV^{-V/A}(A \log\log\log T)^{V/A} \\ &\ll TV^{-V/A}(\log^2 V)^{V/A} \\ &\ll TV^{-V/A}(V^{1/2})^{V/A} \\ &= TV^{-V/2A}, \end{aligned}$$

that is,

$$N_1^{\star}(T, V_1^{\star}) \ll T \exp\left(-\frac{V \log V}{2A}\right). \tag{4.3.9}$$

Finally, we find an upper bound for $N_j(T, V_j)$ for each $2 \leq j \leq \Delta$. For $x^{1/j} \leq T$, Lemma 4.2.2 and Hypothesis H imply that

$$\begin{aligned} \int_T^{2T} |S_j(t)|^{2\ell}\, dt &\ll T\ell! \left( \sum_{p^j \leq x} \frac{|k_1 \Lambda_{\pi_1}(p^j) + \cdots + k_r \Lambda_{\pi_r}(p^j)|^2}{j^2 p^j \log^2 p} \right)^{\ell} \\ &\ll T\big(\ell C_j (k_1^2 + \cdots + k_r^2)\big)^{\ell}, \end{aligned}$$

for each fixed $j$, where $C_j$ is a constant (depending on $j$). Let

$$C_{\max} = \max_{2 \leq j \leq \Delta} C_j$$

be an absolute constant. Then for every $2 \leq j \leq \Delta$, we have

$$\int_T^{2T} |S_j(t)|^{2\ell}\, dt \ll T\big(\ell\, C_{\max} (k_1^2 + \cdots + k_r^2)\big)^{\ell}.$$

Comparing this upper bound to the upper bound for $\int_T^{2T} |S_1^{\star}(t)|^{2\ell} dt$, we conclude that

$$N_j(T, V_j) \ll T \exp\left(-\frac{V \log V}{2A}\right), \tag{4.3.10}$$

for each $2 \leq j \leq \Delta$. By combining (4.3.8), (4.3.9), and (4.3.10), we have that

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V_1^2}{W}\right) + T\exp(-4V\log V) + T\exp\left(-\frac{V\log V}{2A}\right). \quad (4.3.11)$$

We simplify the right-hand side of (4.3.11) by considering each of the three ranges of $V$ specified in the definition of $A$.

**Range 1:** $\sqrt{W} \leq V \leq \frac{W}{B^2}$

Since $V \leq W^2/B^4$, we have

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V_1^2}{W}\right) + T\exp\left(-\frac{V\log V}{2A}\right).$$

By the definition of $V_1$, we have

$$\begin{aligned}
\frac{V}{\sqrt{W}}\exp\left(-\frac{V_1^2}{W}\right) &= \frac{V}{\sqrt{W}}\exp\left(\frac{-V^2}{W}\left(1-\frac{7(B-1)}{4B\log W}\right)^2\right) \\
&= \frac{V}{\sqrt{W}}\exp\left(-\frac{V^2}{W}\left(1-\frac{7(B-1)}{2B\log W}+\frac{49(B-1)^2}{16B^2\log^2 W}\right)\right) \\
&\ll \frac{V}{\sqrt{W}}\exp\left(-\frac{V^2}{W}\left(1-\frac{7(B-1)}{2B\log W}\right)\right) \\
&\ll \frac{V}{\sqrt{W}}\exp\left(\frac{-V^2}{W}\left(1-\frac{4}{\log W}\right)\right).
\end{aligned}$$

Furthermore, in this range we have

$$\frac{V_1^2}{W} \leq \frac{V\log V}{2A} = \frac{V\log V}{B\log W}.$$

It follows that

$$\exp\left(-\frac{V\log V}{2A}\right) \ll \frac{V}{\sqrt{W}}\exp\left(-\frac{V_1^2}{W}\right).$$

Thus we have

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}}\exp\left(-\frac{V^2}{W}\left(1-\frac{4}{\log W}\right)\right)$$

for $\sqrt{W} \leq V \leq \frac{W}{B^2}$.

**Range 2:** $\frac{W}{B^2} \leq V \leq \frac{1}{2B^2} W \log W$

In this range, we have

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}} \exp\left(\frac{-V_1^2}{W}\right) + T\exp\left(\frac{-V\log V}{2A}\right).$$

By the definition of $V_1$, we can estimate the first term as

$$\begin{aligned} T\frac{V}{\sqrt{W}} \exp\left(\frac{-V_1^2}{W}\right) &= T\frac{V}{\sqrt{W}} \exp\left(\frac{-V^2}{W}\left(1-\frac{7B(B-1)V}{4W\log W}\right)^2\right) \\ &\ll T\frac{V}{\sqrt{W}} \exp\left(\frac{-V^2}{W}\left(1-\frac{7B^2V}{4W\log W}\right)^2\right). \end{aligned}$$

Since

$$\frac{V_1^2}{W} \leq \frac{V\log V}{2A}$$

in this range, we have that

$$\exp\left(\frac{-V\log V}{2A}\right) \ll \frac{V}{\sqrt{W}} \exp\left(\frac{-V_1^2}{W}\right).$$

Thus

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}} \exp\left(-\frac{V^2}{W}\left(1-\frac{7B^2V}{4W\log W}\right)^2\right)$$

for $\frac{W}{B^2} \leq V \leq \frac{1}{2B^2} W\log W$.

**Range 3:** $V > \frac{1}{2B^2} W \log W$

In this range, we have

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}} \exp\left(\frac{-V_1^2}{W}\right) + T\exp\left(-4V\log V\right) + T\exp\left(\frac{-V\log V}{2A}\right). \quad (4.3.12)$$

Using the definition of $V_1$, we have

$$\begin{aligned}
\frac{V}{\sqrt{W}} \exp\left(\frac{-V_1^2}{W}\right) &= \frac{V}{\sqrt{W}} \exp\left(\frac{-V^2}{W}\left(1-\frac{7(B-1)}{8B}\right)^2\right) \\
&\leq \frac{V}{\sqrt{W}} \exp\left(\frac{-V^2}{64W}\right) \\
&\ll \exp\left(\frac{-V^2}{64W} + \log V\right) \\
&\ll \exp\left(\frac{-VW\log W}{128B^2 W} + \log V\right) \\
&\ll \exp\left(\frac{-V\log V}{129B^2}\right),
\end{aligned}$$

where we have used the bound $V > \frac{1}{2B^2} W \log W$ in the penultimate step. To estimate the second term on the right-hand side of (4.3.12), note that since $B > 1$, it is true that $\frac{1}{129B^2} \leq 4$. Thus

$$T\exp\left(-4V\log V\right) \ll T\exp\left(\frac{-V\log V}{129B^2}\right).$$

To estimate the third term on the right-hand side of (4.3.12), note that since $B > 1$, it is true that $\frac{1}{129B^2} \leq \frac{1}{2A} = \frac{1}{2B}$. Thus

$$T\exp\left(-\frac{V\log V}{2A}\right) \ll T\exp\left(\frac{-V\log V}{129B^2}\right).$$

Hence,

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll \exp\left(\frac{-V\log V}{129B^2}\right)$$

for all $V > \frac{1}{2B^2} W \log W$. The proposition now follows. □

## 4.4 Proof of Theorem 1.5.4

We now use Proposition 4.3.1 and (4.3.2) to prove Theorem 1.5.4.

*The Proof of Theorem 1.5.4.* We will first show that Proposition 4.3.1 implies that

$$\operatorname{meas}(\mathcal{A}(T,V)) \ll \begin{cases} T(\log T)^{\varepsilon} \exp\left(-\frac{V^2}{W}\right), & \text{if } 3 \le V \le \frac{256W}{B^2}, \\ T(\log T)^{\varepsilon} \exp\left(-\frac{4V}{B^2}\right), & \text{if } V > \frac{256W}{B^2}. \end{cases} \tag{4.4.1}$$

This weaker version of the proposition is enough to establish the theorem. We first consider the case where $3 \le V \le \frac{256W}{B^2}$.

**Range 1:** $\sqrt{W} \le V \le \frac{W}{B^2}$

By Proposition 4.3.1, we have

$$\begin{aligned}
\operatorname{meas}(\mathcal{A}(T,V)) &\ll T\frac{V}{\sqrt{W}} \exp\left(\frac{-V^2}{W}\left(1-\frac{4}{\log W}\right)\right) \\
&\ll T\frac{\sqrt{W}}{B^2} \exp\left(\frac{-V^2}{W}\right) \exp\left(\frac{4V^2}{W\log W}\right) \\
&\ll T\sqrt{W} \exp\left(\frac{-V^2}{W}\right) \exp\left(\frac{4W}{B^4\log W}\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(\frac{-V^2}{W}\right) \exp\left(\log\log T\left(\frac{4(k_1^2+\cdots+k_r^2)}{B^4\log W}\right)\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(\frac{-V^2}{W}\right).
\end{aligned}$$

**Range 2:** $\frac{W}{B^2} \le V \le \frac{256W}{B^2} \le \frac{W\log W}{2B^2}$

By Proposition 4.3.1, we have

$$\begin{aligned}
\operatorname{meas}(\mathcal{A}(T,V)) &\ll T\frac{V}{\sqrt{W}} \exp\left(\frac{-V^2}{W}\left(1-\frac{7B^2V}{4W\log W}\right)^2\right) \\
&\ll T\frac{\sqrt{W}}{B^2} \exp\left(\frac{-V^2}{W}\left(1-\frac{7B^2V}{2W\log W}+\frac{49B^4V^2}{16W^2\log^2 W}\right)\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(\frac{-V^2}{W}\right) \exp\left(\frac{7B^2(256)^3W}{2\log W}-\frac{49W}{16B^4W^3\log^2 W}\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(\frac{-V^2}{W}\right) \exp\left(W\left(\frac{7B^2(256)^3}{2\log W}-\frac{49}{16B^4\log^2 W}\right)\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(\frac{-V^2}{W}\right).
\end{aligned}$$

Since $\log W \to \infty$ and $T \to \infty$, it is not possible to have the case $\frac{W \log W}{2B^2} \leq V \leq \frac{256W}{B^2}$. Thus we have proven (4.4.1) in the range $3 \leq V \leq \frac{256W}{B^2}$.

We now consider the second case where $V > \frac{256W}{B^2}$. First, note that the case $\sqrt{W} \leq \frac{256W}{B^2} \leq V \leq \frac{W}{B^2}$ is not possible.

**Range 2:** $\frac{W}{B^2} \leq \frac{256W}{B^2} \leq V \leq \frac{W \log W}{2B^2}$

By Proposition 4.3.1, we have

$$\begin{aligned}
\operatorname{meas}(\mathcal{A}(T,V)) &\ll T\frac{V}{\sqrt{W}} \exp\left(\frac{-V^2}{W}\left(1 - \frac{7B^2V}{4W\log W}\right)^2\right) \\
&\ll T\frac{W\log W}{2B^2\sqrt{W}} \exp\left(\frac{-256W}{B^2W}V\left(1 - \frac{7B^2W\log W}{8B^2W\log W}\right)^2\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(\frac{-256}{64B^2}V\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(-\frac{4V}{B^2}\right).
\end{aligned}$$

**Range 3:** $\frac{W \log W}{2B^2} \leq \frac{256W}{B^2} \leq V$

By Proposition 4.3.1, we have

$$\begin{aligned}
\operatorname{meas}(\mathcal{A}(T,V)) &\ll T\exp\left(\frac{-1}{129B^2}V\log V\right) \\
&\ll T(\log T)^{\varepsilon} \exp\left(-\frac{4V}{B^2}\right).
\end{aligned}$$

We now use these cruder bounds on $\operatorname{meas}(\mathcal{A}(T,V))$ to prove Theorem 1.5.4. Write

$$\int_{-\infty}^{\infty} \exp(2V)\operatorname{meas}(\mathcal{A}(T,V))dV = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{-\infty}^{3} \exp(2V)\operatorname{meas}(\mathcal{A}(T,V))dV, \qquad I_2 = \int_{3}^{256W/B^2} \exp(2V)\operatorname{meas}(\mathcal{A}(T,V))dV,$$

and

$$I_3 = \int_{256W/B^2}^{\infty} \exp(2V)\,\mathrm{meas}(\mathcal{A}(T,V))dV.$$

By definition, $\mathrm{meas}(\mathcal{A}(T,V)) \le T$ and thus $I_1 \ll T$. Applying the bounds for $\mathrm{meas}(\mathcal{A}(T,V))$ given in (4.4.1) to $I_2$ and $I_3$, it follows that

$$\begin{aligned} I_2 &\ll T(\log T)^{\varepsilon} \int_3^{256W/B^2} \exp\left(2V - \frac{V^2}{W}\right) dV \\ &\ll T(\log T)^{\varepsilon} e^{W} 256W \\ &\ll T(\log T)^{k_1^2+\cdots k_r^2+\varepsilon}, \end{aligned}$$

and similarly that $I_3 \ll T(\log T)^{\varepsilon}$. Consequently, these estimates imply that

$$\int_T^{2T} |L(\tfrac{1}{2}+it,\pi_1)|^{2k_1} \cdots |L(\tfrac{1}{2}+it,\pi_r)|^{2k_r} dt = 2\big(I_1+I_2+I_3\big) \ll T(\log T)^{k_1^2+\cdots+k_r^2+\varepsilon}.$$

Theorem 1.5.4 now follows by summing over the dyadic intervals $[\frac{T}{2},T], [\frac{T}{4},\frac{T}{2}], [\frac{T}{8},\frac{T}{4}], \ldots$. $\square$

# 5 THE PROOF OF THEOREM 1.5.6

In this section, we modify the proof of Theorem 1.5.4 to deduce Theorem 1.5.6. Throughout this section, let $K$ be a finite extension of $\mathbb{Q}$, and let $\zeta_K(s)$ be the associated Dedekind zeta-function. As before, our starting point is the observation that

$$\int_T^{2T} |\zeta_K(\tfrac{1}{2}+it)|^{2k}\, dt = 2\int_{-\infty}^{\infty} \exp(2V)\,\mathrm{meas}(\mathcal{K}(T,V))\, dV \tag{5.0.1}$$

where

$$\mathcal{K}(T,V) = \{t \in [T,2T] : k\log|\zeta_K(\tfrac{1}{2}+it)| \geq V\}.$$

In order to bound the measure of $\mathcal{K}(T,V)$, we need analogues of Lemma 4.2.3 and (4.3.3) for $\zeta_K(s)$.

## 5.1 Lemmas

For $\Re(s) > 1$, define

$$\frac{\zeta_K'}{\zeta_K}(s) := \frac{d}{ds}\log\zeta_K(s) = -\sum_{n=1}^{\infty}\frac{\Lambda_K(n)}{n^s}.$$

Since $\zeta_K(s)$ satisfies the Ramanujan-Petersson conjecture, we have

$$|\Lambda_K(n)| \leq [K:\mathbb{Q}]\Lambda(n). \tag{5.1.1}$$

Then the following analogue of Lemma 4.2.3 holds.

**Lemma 5.1.1.** *Let $\lambda_0 = 0.4912\ldots$ denote the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$. Then, assuming the generalized Riemann hypothesis for $\zeta_K(s)$, for all $\lambda_0 \le \lambda \le \log x/2$ and $\log x \ge 2$, we have*

$$\log|\zeta_K(\tfrac{1}{2}+it)| \le \Re \sum_{n\le x} \frac{\Lambda_K(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}+it}\log n} \frac{\log x/n}{\log x} + \frac{(1+\lambda)}{2}\frac{[K:\mathbb{Q}]\log T}{\log x} + O\Big(\frac{1}{\log x}\Big)$$

*for $T \le t \le 2T$ and $T$ sufficiently large, where the implied constant in the error term depends only on $K$.*

*Proof.* This is a consequence of Theorem 2.1 of Chandee [12]. $\square$

The analogue of (4.3.3) follows from the Chebotarev density theorem.

**Lemma 5.1.2.** *Let $K$ be a finite Galois extension of $\mathbb{Q}$, and let $p$ denote a rational prime. Then*

$$\sum_{p\le x} r_K(p)^2 \sim [K:\mathbb{Q}] \sum_{p\le x} 1$$

*as $x \to \infty$, and in particular*

$$\sum_{p\le x} \frac{r_K(p)^2}{p} \sim [K:\mathbb{Q}]\log\log x. \tag{5.1.2}$$

*Proof.* Let $(p)$ denote the principal ideal in $\mathcal{O}_K$ generated by $p$. Then

$$(p) = \mathfrak{P}_1^{e_1}\cdots\mathfrak{P}_r^{e_r},$$

where the $\mathfrak{P}_i$ are the distinct prime ideals in $\mathcal{O}_K$ lying above $p$ with norm $p^{f_i}$. It follows that

$$\sum_{i=1}^{r} e_i f_i = [K:\mathbb{Q}].$$

If $p$ is unramified in $K$, then $e_1 = \cdots = e_r = 1$. Since $K$ is Galois, all the $\mathfrak{P}_i$ lying above $p$ are conjugate. Thus $f_1 = \cdots = f_r = f$, say. Therefore, for unramified primes $p$, we see that $r_K(p) \neq 0$ if and only if $f = 1$. In this case, $p$ completely splits, $r = [K:\mathbb{Q}]$, and hence $r_K(p) = [K:\mathbb{Q}]$. That is, for unramified primes $p$, we have

$$r_K(p) = \begin{cases} [K:\mathbb{Q}], & \text{if and only if } p \text{ splits completely}, \\ 0, & \text{otherwise}. \end{cases}$$

Since there are only a finite number of ramified primes, it follows that

$$\sum_{p \leq x} r_K(p)^2 \;=\; \sum_{\substack{p \leq x \\ p \text{ unramified}}} r_K(p)^2 + O(1) \;=\; \sum_{\substack{p \leq x \\ p \text{ splits completely}}} [K:\mathbb{Q}]^2 + O(1).$$

On the other hand, the Chebotarev density theorem implies that

$$\sum_{\substack{p \leq x \\ p \text{ splits completely}}} 1 \;\sim\; \frac{1}{[K:\mathbb{Q}]} \sum_{p \leq x} 1,$$

as $x \to \infty$. Thus,

$$\sum_{p \leq x} r_K(p)^2 \sim [K:\mathbb{Q}] \sum_{p \leq x} 1,$$

proving the first assertion of the lemma. By the prime number theorem, we have

$$\sum_{p \leq x} r_K(p)^2 \sim \frac{[K:\mathbb{Q}]x}{\log x}$$

as $x \to \infty$. The claim in (5.1.2) now follows by partial summation. $\square$

We note here that in order to prove Theorem 1.5.6, it is not necessary to derive an asymptotic formula for the sum in (5.1.2). An upper bound of the form

$$\sum_{p \leq x} \frac{r_K(p)^2}{p} \leq [K:\mathbb{Q}] \log\log x + O(1)$$

for the sum in (5.1.2) would be sufficient and is more easily derived. For instance, since $0 \le r_K(p) \le [K:\mathbb{Q}]$, we see that

$$\sum_{p\le x} \frac{r_K(p)^2}{p} \le [K:\mathbb{Q}] \sum_{p\le x} \frac{r_K(p)}{p} \le [K:\mathbb{Q}] \log\log x + O(1)$$

by Landau's Prime Ideal Theorem.

## 5.2 The Frequency of Large Values of $|\zeta_K(\frac{1}{2}+it)|$

Define the set

$$\mathcal{K}(T,V) = \{t \in [T,2T] : k \log|\zeta_K(\tfrac{1}{2}+it)| \ge V\},$$

and choose

$$W = k^2[K:\mathbb{Q}]\log\log T, \qquad B = k[K:\mathbb{Q}]+1, \qquad \text{and} \qquad \Delta = 2,$$

since the Ramanujan-Petersson conjecture holds for $\zeta_K(s)$. Note that $B > 1$ for all $k > 0$. We prove estimates for the size of $\mathcal{K}(T,V)$ using the previous lemmas.

**Proposition 5.2.1.** *Let $K$ be a Galois extension of $\mathbb{Q}$. Assume the generalized Riemann hypothesis for $\zeta_K(s)$. Then, for sufficiently large $T$, if $\sqrt{W} \le V \le \frac{W^2}{B}$, we have*

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}} \exp\left(-\frac{V^2}{W}\left(1-\frac{4}{\log W}\right)\right);$$

*if $\frac{W}{B^2} \le V \le \frac{1}{2B^2} W\log W$, we have*

$$\operatorname{meas}(\mathcal{S}(T,V)) \ll T\frac{V}{\sqrt{W}} \exp\left(-\frac{V^2}{W}\left(1-\frac{7B^2V}{4W\log W}\right)^2\right);$$

*and if* $\frac{1}{2B^2} W \log W \leq V$ *, we have*

$$\mathrm{meas}(\mathcal{S}(T,V)) \ll T \exp\left(\frac{-1}{129B^2} V \log V\right)$$

*for any* $k > 0$ *when* $T$ *is sufficiently large. The implied constants in each range depend on* $k$ *and* $[K:\mathbb{Q}]$.

*Proof.* The proof is analogous to the proof of Proposition 4.3.1. Let $K$ be a Galois extension of $\mathbb{Q}$, and let $\zeta_K(s)$ denote the associated Dedekind zeta-function. Let $\lambda = \lambda_0 < 1/2$ and $\varepsilon < (2-4\lambda_0)/3$. Choosing $x = (\log T)^{2-\varepsilon}$, it follows from Lemma 5.1.1 and (5.1.1) that

$$\begin{aligned}\log|\zeta_K(\tfrac{1}{2}+it)| &\leq [K:\mathbb{Q}](\log T)^{1-\varepsilon} + \frac{(1+\lambda_0)[K:\mathbb{Q}]\log T}{2(2-\varepsilon)\log\log T} + O\left(\frac{1}{(2-\varepsilon)\log\log T}\right)\\ &\leq \frac{3[K:\mathbb{Q}]}{8}\frac{\log T}{\log\log T}\end{aligned}$$

for sufficiently large $T$. Therefore, we see that

$$k\log|\zeta_K(\tfrac{1}{2}+it)| \leq \frac{3k[K:\mathbb{Q}]}{8}\frac{\log T}{\log\log T}$$

when $T$ is large. We can thus assume that

$$\sqrt{W} \leq V \leq \frac{3(B-1)}{8}\frac{\log T}{\log\log T} \tag{5.2.1}$$

while proving the proposition. As in the proof of Proposition 4.3.1, define a parameter, $A$, as

$$A = \begin{cases} \dfrac{B}{2}\log W, & \text{if } \sqrt{W} \leq V \leq \frac{W}{B^2},\\[2ex] \dfrac{1}{2BV} W\log W, & \text{if } \frac{W}{B^2} < V \leq \frac{1}{2B^2}W\log W,\\[2ex] B, & \text{if } V > \frac{1}{2B^2}W\log W,\end{cases}$$

and let $x = T^{A/V}$ and $z = x^{1/\log\log T}$. Choosing $\lambda = 1/2$ in Lemma 5.1.1, we have

$$k \log|\zeta_K(\tfrac{1}{2} + it)| \le |S_1(t)| + |S_1^\star(t)| + |S_2(t)| + \frac{3(B-1)}{4}\frac{V}{A} + O(1),$$

where

$$S_1(t) = \sum_{p \le z} \frac{k\Lambda_K(p)}{p^{\frac{1}{2} + \frac{\lambda}{\log x} + it} \log p} \frac{\log(x/p)}{\log x},$$

$$S_1^\star(t) = \sum_{z < p \le x} \frac{k\Lambda_K(p)}{p^{\frac{1}{2} + \frac{\lambda}{\log x} + it} \log p} \frac{\log(x/p)}{\log x},$$

and

$$S_2(t) = \sum_{p^2 \le x} \frac{k\Lambda_K(p)}{p^{1 + \frac{2\lambda}{\log x} + 2it} \log p^j} \frac{\log(x/p^j)}{\log x}.$$

Since the Ramanujan-Petersson Conjecture is true in this case, the contribution from the prime powers $p^j$ for which $j \ge 3$ is $O(1)$. Indeed, (5.1.1) gives

$$\sum_{\substack{p^j \le x \\ j \ge 3}} \frac{|k\Lambda_K(p^j)|}{p^{j(\frac{1}{2} + \frac{\lambda}{\log x} + it)} \log p^j} \frac{\log(x/p^j)}{\log x} \ll \sum_{\substack{p^j \le x \\ j \ge 3}} \frac{|k\Lambda_K(p^j)|}{j p^{j/2} \log p} \ll 1.$$

Let

$$V_1 := V\left(1 - \frac{7(B-1)}{8A}\right), \quad \text{and} \quad V_1^\star = V_2 := \frac{(B-1)V}{16A}.$$

Note that if $t \in \mathcal{K}(T, V)$, then at least one of the following inequalities holds:

$$|S_1^\star(t)| \ge V_1^\star \quad \text{or} \quad |S_j(t)| \ge V_j$$

for either $j = 1$ or $j = 2$. If we define

$$N_j(T, V_j) := \operatorname{meas}\{t \in [T, 2T] : |S_j(t)| \ge V_j\}$$

for $j = 1, 2$ and $N_1^\star(T, V_1^\star)$ similarly, we can bound $N_j(T, V_j)$ and $N_1^\star(T, V_1^\star)$ using Lemma 4.2.2 since

$$N_j(T, V_j) \leq (V_j)^{-2\ell} \int_T^{2T} |S_j(t)|^{2\ell} dt$$

and

$$N_1^\star(T, V_1^\star) \leq (V_1^\star)^{-2\ell} \int_T^{2T} |S_1^\star(t)|^{2\ell} dt$$

for any non-negative integer $\ell$.

Let us first estimate $N_1(T, V_1)$. Letting $\ell$ be any natural number such that $\ell \leq \frac{V \log\log T}{A}$, Lemma 4.2.2 and (5.1.2) together with the same reasoning in the proof of Proposition 4.3.1 imply that

$$\int_T^{2T} |S_1(t)|^{2\ell} dt \ll T\sqrt{\ell} \left( \frac{\ell W}{e} \right)^\ell,$$

Thus we have

$$N_1(T, V_1) \ll T\sqrt{\ell} \left( \frac{\ell W}{e V_1^2} \right)^\ell. \tag{5.2.2}$$

We consider separately the two cases where $V \leq \frac{W^2}{B^4}$ and $V > \frac{W^2}{B^4}$. In the first case, we choose $\ell = \lfloor \frac{V_1^2}{W} \rfloor$ in (5.2.2) and find that

$$N_1(T, V_1) \ll T \frac{V}{\sqrt{W}} \exp\left( -\frac{V_1^2}{W} \right).$$

In the case where $V > \frac{W^2}{B^4}$, we choose $\ell = \lfloor 10V \rfloor$ in (5.2.2) and find that

$$N_1(T, V_1) \ll T \exp(-4V \log V).$$

Hence

$$N_1(T, V_1) \ll T \frac{V}{\sqrt{W}} \exp\left( -\frac{V_1^2}{W} \right) + T \exp(-4V \log V) \tag{5.2.3}$$

for all $V$.

Next, we find an upper bound for $N_1^\star(T, V_1^\star)$. Lemma 4.2.2 and (5.1.2) imply that for sufficiently large $T$ we have

$$\int_T^{2T} |S_1^\star(t)|^{2\ell} dt \ll T \left\{ 2\ell k^2 [K : \mathbb{Q}] \log\log\log T \right\}^\ell,$$

for any natural number $\ell \leq \frac{V}{A}$ since $x = T^{A/V}$. Choosing $\ell = \lfloor \frac{V}{A} \rfloor$, we have that

$$N_1^\star(T, V_1^\star) \ll T \exp\left( -\frac{V \log V}{2A} \right). \tag{5.2.4}$$

Finally, we find an upper bound on $N_2(T, V_2)$. First, note that

$$\sum_p \frac{1}{p^2} < \frac{1}{2}.$$

Since $\zeta_K(s)$ satisfies the Ramanujan-Petersson Conjecture, by Lemma 4.2.2 we have that

$$\begin{aligned} \int_T^{2T} |S_2(t)|^{2\ell} dt &\ll T\ell! \Big( \sum_{p \leq \sqrt{x}} \frac{|k\Lambda_K(p^2)|^2}{4p^2 \log^2 p} \Big)^\ell \\ &\ll T \left\{ \ell k^2 [K : \mathbb{Q}]^2 \right\}^\ell. \end{aligned}$$

Comparing this upper bound to the upper bound for $\int_T^{2T} |S_1^\star(t)|^{2\ell} dt$, we conclude that

$$N_2(T, V_2) \ll T \exp\Big( -\frac{V}{2A} \log V \Big). \tag{5.2.5}$$

The proposition now follows upon combining (5.2.3), (5.2.4), and (5.2.5) and simplifying. $\square$

## 5.3 The Proof of Theorem 1.5.6

We now use Proposition 5.2.1 and (5.0.1) to prove Theorem 1.5.6.

*Proof.* Proposition 5.2.1 implies that

$$\mathrm{meas}(\mathcal{K}(T,V)) \ll \begin{cases} T(\log T)^{\varepsilon} \exp\left(-\frac{V^2}{W}\right), & \text{if } 3 \leq V \leq \frac{256W}{B^2}, \\ T(\log T)^{\varepsilon} \exp\left(-\frac{4V}{B^2}\right), & \text{if } V > \frac{256W}{B^2}. \end{cases}$$

Inserting these bounds into (5.0.1) and estimating the range $V < 3$ trivially, we deduce Theorem 1.5.6. $\square$

# 6 THE PROOFS OF THEOREM 1.5.7 AND THEOREM 1.5.8

In this chapter we prove Theorem 1.5.7 and Theorem 1.5.8. Both results were proved in collaboration with Micah B. Milinovich and appear in [63]. We first collect some well-known results from complex analysis that will be used in both proofs.

## 6.1 Preliminary Results

We will make use of the following two results, which can be found in Chapter 5 of the book by Montgomery and Vaughan [69].

**Perron's Formula.** *Let $a_n$ be an arithmetic function, and let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be the corresponding Dirichlet series absolutely convergent for $\Re(s) > \sigma_c$. If $\sigma_0 > \max(0, \sigma_c)$ and $x > 0$, then*

$$\sum_{n \leq x}{}^{\flat} a_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} ds.$$

*Here $\sum^{\flat}$ indicates that if $x$ is an integer, then the last term is to be counted with weight* $1/2$.

**Plancherel's Theorem.** *Let $w(x)$ be a weight function. Suppose that $\int_0^\infty |w(x)| x^{-\sigma-1} dx < \infty$, and also that $\int_0^\infty |w(x)|^2 x^{-2\sigma-1} dx < \infty$. Put $K(s) = \int_0^\infty w(x) x^{-s-1} dx$. Then*

$$\int_0^\infty |w(x)|^2 x^{-2\sigma-1} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |K(\sigma + it)|^2 dt.$$

## 6.2 The Proof of Theorem 1.5.7

We follow the proof of Theorem 1 of Selberg [80], who studied the distribution of primes in short intervals using upper bounds for moments of the logarithmic derivative of $\zeta(s)$ near the critical line. (See also Section 4 of Goldston, Gonek, and Montgomery [33].)

*Proof.* For $K$ a finite Galois extension of $\mathbb{Q}$, let

$$c_K = \frac{2^{r_1}(2\pi)^{r_2} hR}{w\sqrt{|d|}}, \quad S(x) = \sum_{n \le x} r_K(n), \quad \text{and} \quad S_0(x) = \frac{1}{2}\lim_{\varepsilon \to 0}\Big(S(x+\varepsilon) + S(x-\varepsilon)\Big)$$

so that $S(x) = S_0(x)$ for almost all $x$. Perron's Formula implies that

$$S_0(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \zeta_K(s)\frac{x^s}{s}\, ds. \tag{6.2.1}$$

The Generalized Lindelöf Hypothesis for $\zeta_K(s)$ in $t$-aspect is the statement that

$$\zeta_K(\tfrac{1}{2} + it) \ll t^{\varepsilon},$$

where the implied constant depends on $K$. Assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, we move the contour in (6.2.1) left from $\Re(s) = 2$ to $\Re(s) = 1/2$ passing over a pole of the integrand at $s = 1$ and no other singularities. Here we are implicitly using the Generalized Lindelöf Hypothesis for $\zeta_K(s)$ in $t$-aspect, which follows from the Generalized Riemann Hypothesis, to justify the contour shift. Thus by the residue calculation in (1.3.1) and a variable change, we have

$$S_0(x) - c_K x = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \zeta_K(s)\frac{x^s}{s}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta_K(\tfrac{1}{2}+it)\left(\frac{x^{\frac{1}{2}+it}}{\frac{1}{2}+it}\right) dt.$$

Applying this formula twice with the values $x = e^{\tau+\kappa}$ and $x = e^{\tau}$ and then differencing, it follows that

$$\frac{S_0(e^{\kappa+\tau})-S_0(e^{\tau})-c_K(e^{\kappa}-1)e^{\tau}}{e^{\tau/2}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta_K(\tfrac{1}{2}+it)\left(\frac{e^{\kappa\left(\frac{1}{2}+it\right)}-1}{\frac{1}{2}+it}\right)e^{i\tau t}\,dt. \tag{6.2.2}$$

Note that the left-hand side of (6.2.2) is a Fourier transform (appropriately normalized) of

$$\zeta_K(\tfrac{1}{2}+it)\left(\frac{e^{\kappa\left(\frac{1}{2}+it\right)}-1}{\frac{1}{2}+it}\right),$$

so (6.2.2) gives a Fourier transform relation for all $\tau \in \mathbb{R}$. By Plancherel's Theorem, since $S_0(x) = S(x)$ almost everywhere, we have

$$\int_{-\infty}^{\infty} \left|S(e^{\kappa+\tau})-S(e^{\tau})-c_K(e^{\kappa}-1)e^{\tau}\right|^2 \frac{d\tau}{e^{\tau}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\zeta_K(\tfrac{1}{2}+it)\right|^2 \left|\frac{e^{\kappa\left(\frac{1}{2}+it\right)}-1}{\frac{1}{2}+it}\right|^2 dt.$$

Observing that the integrand on the left-hand side is even and letting $x = e^{\tau}, X \geq T \geq 2$, and $e^{\kappa} = 1 + 1/T$, we derive that

$$\begin{aligned}
\int_X^{2X} \left|S\left(x+\frac{x}{T}\right)-S(x)-c_K\frac{x}{T}\right|^2 \frac{dx}{x^2} &\leq \int_0^{\infty} \left|S\left(x+\frac{x}{T}\right)-S(x)-c_K\frac{x}{T}\right|^2 \frac{dx}{x^2} \\
&= \frac{1}{\pi}\int_0^{\infty} \left|\zeta_K(\tfrac{1}{2}+it)\right|^2 \left|\frac{e^{\kappa\left(\frac{1}{2}+it\right)}-1}{\frac{1}{2}+it}\right|^2 dt \\
&= \frac{1}{\pi}\sum_{\ell=0}^{\infty}\int_{(2^{\ell}-1)T}^{(2^{\ell+1}-1)T} \left|\zeta_K(\tfrac{1}{2}+it)\right|^2 \left|\frac{e^{\kappa\left(\frac{1}{2}+it\right)}-1}{\frac{1}{2}+it}\right|^2 dt \\
&\ll \sum_{\ell=0}^{\infty}\frac{1}{(2^{\ell}T)^2}\int_0^{(2^{\ell+1}-1)T} \left|\zeta_K(\tfrac{1}{2}+it)\right|^2 dt.
\end{aligned}$$

It follows from this and Theorem 1.5.6 that

$$\frac{1}{X^2}\int_X^{2X} \left|S\left(x+\frac{x}{T}\right)-S(x)-c_K\frac{x}{T}\right|^2 dx \ll \sum_{\ell=0}^{\infty}\frac{1}{2^{\ell}T}(\log T)^{[K:\mathbb{Q}]+\varepsilon} \ll \frac{(\log T)^{[K:\mathbb{Q}]+\varepsilon}}{T}$$

for any $\varepsilon > 0$. Theorem 1.5.7 now follows by choosing $y = x/T$. □

## 6.3 The Proof of Theorem 1.5.8

We now indicate how to modify the proof of Theorem 1.5.7 to obtain Theorem 1.5.8. The proof given here is an expanded version of the argument given in [63].

*Proof.* For $k \geq 0$ an integer and $k_1, \ldots, k_r \in \mathbb{N}$, let

$$L(s) = \zeta(s)^k \prod_{j=1}^{r} L(s, \pi_j)^{k_j} \tag{6.3.1}$$

be an automorphic $L$-function and let each $L(s, \pi_j)$ satisfy the conditions of Theorem 1.5.4. For $\Re(s) > 1$, we set

$$L(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{a_L(n)}{n^s}, & \text{if } k = 0, \\ \displaystyle\sum_{n=1}^{\infty} \frac{b_L(n)}{n^s}, & \text{if } k \in \mathbb{N}. \end{cases}$$

and for $x > 0$, we define

$$R_L(x) = \operatorname*{Res}_{s=1} \left( L(s) \frac{x^s}{s} \right).$$

If $k = 0$, we see that $L(s)$ is entire and denote

$$A_L(x) = \sum_{n \leq x} a_L(n).$$

If $k \in \mathbb{N}$, we see that $L(s)$ has a pole at $s = 1$ and denote

$$B_L(x) = \sum_{n \leq x} b_L(n).$$

Furthermore, let

$$C(x) = \begin{cases} \dfrac{1}{2} \lim\limits_{\varepsilon \to 0} \big( A_L(x+\varepsilon) - A_L(x-\varepsilon) \big), & \text{if } k = 0, \\ \\ \dfrac{1}{2} \lim\limits_{\varepsilon \to 0} \big( B_L(x+\varepsilon) - B_L(x-\varepsilon) \big), & \text{if } k \in \mathbb{N}, \end{cases}$$

so that $C(x) = A_L(x)$ (or $B_L(x)$) for almost all $x$. Perron's Formula and (6.3.1) imply that

$$C(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(s) \frac{x^s}{s} ds.$$

Moving the line of integration left from $\Re(s) = 2$ to $\Re(s) = 1/2$, we have

$$C(x) - R_L(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} L(s) \left( \frac{x^{\frac{1}{2}+it}}{\frac{1}{2}+it} \right) dt.$$

Notice that if $k = 0$, then $R_L(s) = 0$ because there is no pole. Applying this formula twice with the values $x = e^{\tau+\kappa}$ and $x = e^{\tau}$ and then differencing, it follows that

$$\frac{C(e^{\kappa+\tau}) - C(e^{\tau}) - R_L(e^{\kappa+\tau}) + R_L(e^{\tau})}{e^{\tau/2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} L(\tfrac{1}{2}+it) \left( \frac{e^{\kappa\left(\frac{1}{2}+it\right)} - 1}{\frac{1}{2}+it} \right) e^{it\tau} dt \qquad (6.3.2)$$

for all $\tau \in \mathbb{R}$ and all $\kappa \geq 0$. As in the proof of Theorem 1.5.6, we next apply Plancherel's Theorem to 6.3.2. If $k = 0$, then since $C(x) = A_L(x)$ almost everywhere, we find that

$$\int_{-\infty}^{\infty} \frac{|A_L(e^{\kappa+\tau}) - A_L(e^{\tau})|^2}{e^{\tau}} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|L(\tfrac{1}{2}+it)\right|^2 \left| \frac{e^{\kappa\left(\frac{1}{2}+it\right)} - 1}{\frac{1}{2}+it} \right|^2 dt.$$

If $k \in \mathbb{N}$, then since $C(x) = B_L(x)$ almost everywhere, we have

$$\int_{-\infty}^{\infty} \frac{|B_L(e^{\kappa+\tau}) - B_L(e^{\tau}) - R_L(e^{\kappa+\tau}) + R_L(e^{\tau})|^2}{e^{\tau}} d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|L(\tfrac{1}{2}+it)\right|^2 \left| \frac{e^{\kappa\left(\frac{1}{2}+it\right)} - 1}{\frac{1}{2}+it} \right|^2 dt.$$

In both cases, the integrand on the left-hand side is even. Therefore, letting $x = e^{\tau}, X \geq T \geq 2$, and $e^{\kappa} = 1 + 1/T$, by Theorem 1.5.4 we deduce that

$$\frac{1}{X^2}\int_X^{2X}\left|\sum_{x<n\leq x+y} a_L(n)\right|^2 dx \ll \frac{(\log T)^{k_1^2+\cdots+k_r^2+\varepsilon}}{T}$$

and

$$\frac{1}{X^2}\int_X^{2X}\left|\sum_{x<n\leq x+y} b_L(n) - \Big(R_L(x+y) - R_L(x)\Big)\right|^2 dx \ll \frac{(\log T)^{k_1^2+\cdots+k_r^2+\varepsilon}}{T}$$

for any $\varepsilon > 0$. Theorem 1.5.8 now follows by choosing $y = x/T$.

$\square$

# 7 THE PROOF OF THEOREM 1.6.1

In this chapter, we prove Theorem 1.6.1, which is joint work with Micah B. Milinovich and also appears in [63]. In this case, we need to understand the behavior of the Dirichlet series coefficients of automorphic $L$-functions averaged over the squares of primes. Let $L(s,\pi)$ be an $L$-function attached to a self-contragredient irreducible cuspidal automorphic representation $\pi$ of $\mathrm{GL}(m)$ over $\mathbb{Q}$ (i.e. $\pi = \widetilde{\pi}$). The Rankin-Selberg $L$-function $L(s,\pi\otimes\widetilde{\pi}) = L(s,\pi\otimes\pi)$ factors as the product of the symmetric and exterior square $L$-functions

$$L(s,\pi\otimes\widetilde{\pi}) = L(s,\pi,\vee^2)\cdot L(s,\pi,\wedge^2)$$

and has a simple pole at $s=1$, see Bump and Ginzberg [11]. This pole must be carried by one of the factors on the right-hand side. Following [78], we denote the order of the pole of $L(s,\pi,\wedge^2)$ as $(1+\delta(\pi))/2$. Then it is known that

$$\sum_{p\le x}\Lambda_\pi(p^2) \sim -\delta(\pi)\, x \qquad (7.0.1)$$

as $x\to\infty$. In contrast to the proof of Theorem 1.5.4, we must assume the Ramanujan-Petersson Conjecture in the proof of Theorem 1.6.1 to handle the contribution from the prime squares. We restate the Ramanujan-Petersson Conjecture here for convenience.

**Ramanujan-Petersson Conjecture.** The local parameters $\alpha_j(p)$ in (1.3.2) satisfy $|\alpha_j(p)| = 1$ for all but a finite number of primes $p$.

As in the proof of Theorem 1.5.4, the starting point is the observation that

$$\begin{aligned}\sum_{|d|\le X}^{\flat} L(\tfrac{1}{2},\pi_1\otimes\chi_d)^{k_1}\cdots L(\tfrac{1}{2},\pi_r\otimes\chi_d)^{k_r}\\ =\int_{-\infty}^{\infty}\exp\left(V-\frac{k_1\delta(\pi_1)+\cdots+k_r\delta(\pi_r)}{2}\log\log X\right)\mathcal{N}(X,V)\,dV,\end{aligned}\tag{7.0.2}$$

where $\mathcal{N}(X,V)$ denotes the number of fundamental discriminants $d$ with $|d|\le X$ such that

$$k_1\log\left|L(\tfrac{1}{2},\pi_1\otimes\chi_d)\right|+\cdots+k_r\log\left|L(\tfrac{1}{2},\pi_r\otimes\chi_d)\right|\ge V-\left(\frac{k_1\delta(\pi_1)+\cdots+k_r\delta(\pi_r)}{2}\right)\log\log X.\tag{7.0.3}$$

We can bound $\mathcal{N}(X,V)$ with the following analogues of Lemmas 4.2.2 and 4.2.3. (Note that the definition of $\mathcal{N}(X,V)$ takes into account the contribution from the squares of primes in Lemma 7.1.2, below.)

## 7.1 Lemmas

**Lemma 7.1.1.** *Let $X$ and $y$ be real numbers and $\ell$ be a natural number with $y^\ell\le X^{1/2}/\log X$. For any complex numbers $b(p)$ we have*

$$\sum_{|d|\le X}^{\flat}\left|\sum_{2<p\le y}\frac{b(p)\chi_d(p)}{p^{1/2}}\right|^{2\ell}\ll X\frac{(2\ell)!}{\ell!2^\ell}\left(\sum_{p\le y}\frac{|b(p)|^2}{p}\right),$$

*where the implied constant is absolute.*

*Proof.* This is Lemma 6.3 of Soundararajan and Young [86]. □

**Lemma 7.1.2.** *Let $L(s,\pi)$ be an $L$-function attached to an irreducible cuspidal automorphic representation $\pi$ on $\mathrm{GL}(m)$ over $\mathbb{Q}$, and let $d$ be a fundamental discriminant. Let $\lambda_0=0.4912\ldots$ denote the unique positive real number satisfying $e^{-\lambda_0}=\lambda_0+\lambda_0^2/2$. Then, assuming the Generalized Riemann Hypothesis for $L(s,\pi\otimes\chi_d)$, for all $\lambda_0\le\lambda\le\log x/2$ and $\log x\ge 2$,* *we have*

$$\log\left|L(\tfrac{1}{2},\pi\otimes\chi_d)\right| \le \Re\left\{\sum_{n\le x}\frac{\Lambda_\pi(n)\chi_d(n)}{n^{\frac{1}{2}+\frac{\lambda}{\log x}}\log n}\frac{\log x/n}{\log x}\right\} + \frac{(1+\lambda)}{2}\frac{m\log|d|}{\log x} + O\Big(\frac{1}{\log x}\Big),$$

*where the implied constant depends only on* $\pi$.

*Proof.* This follows from Theorem 2.1 of Chandee [12]. □

**Lemma 7.1.3.** *Let* $L(s,\pi)$ *be an* $L$*-function attached to an irreducible cuspidal automorphic representation* $\pi$ *on* $\mathrm{GL}(m)$ *over* $\mathbb{Q}$*, and let* $d$ *be a fundamental discriminant. Then, assuming the Ramanujan-Petersson Conjecture, we have that*

$$\sum_{p^2\le x}\frac{\Lambda_\pi(p^2)\chi_d(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} = -\frac{\delta(\pi)}{2}\log\log x + O(\log\log\log|d|).$$

*Proof.* Note that

$$\begin{aligned}\sum_{p^2\le x}\frac{\Lambda_\pi(p^2)\chi_d(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} &= \sum_{\substack{p^2\le x\\ p\nmid d}}\frac{\Lambda_\pi(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2}\\ &= \sum_{p^2\le x}\frac{\Lambda_\pi(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} - \sum_{\substack{p^2\le x\\ p\mid d}}\frac{\Lambda_\pi(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2}.\end{aligned}$$

By (7.0.1) and partial summation, the first sum is

$$\sum_{p^2\le x}\frac{\Lambda_\pi(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} \sim -\frac{\delta(\pi)}{2}\log\log x.$$

If $\Lambda_\pi(p^2)>0$ for all $p|d$, then we conclude

$$\sum_{p^2\le x}\frac{\Lambda_\pi(p^2)\chi_d(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} = -\frac{\delta(\pi)}{2}\log\log x + O(1).$$

More care is required if $\Lambda_\pi(p^2) < 0$ for some $p|d$. Let $\omega(n)$ denote the number of distinct prime divisors of $n$. Then, by Merten's Theorem, we have

$$\begin{aligned}\sum_{p|d}\frac{1}{p} &\le \sum_{p\le\log|d|}\frac{1}{p} + \sum_{\substack{\log|d|\le p\\ p|d}}\frac{1}{p}\\ &\le \sum_{p\le\log|d|}\frac{1}{p} + \frac{1}{\log|d|}\omega(d)\\ &= O(\log\log\log|d|) + O\left(\frac{1}{\log|d|}\frac{\log|d|}{\log\log|d|}\right)\\ &= O(\log\log\log|d|).\end{aligned}$$

Thus, assuming the Ramanujan-Petersson Conjecture, it follows that

$$-\sum_{\substack{p^2\le x\\ p|d}}\frac{\Lambda_\pi(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} \ll \frac{m}{2}\sum_{p|d}\frac{1}{p} \ll \log\log\log|d|,$$

In either case, we conclude that

$$\sum_{p^2\le x}\frac{\Lambda_\pi(p^2)\chi_d(p^2)}{p^{1+\frac{2\lambda}{\log x}}\log p^2} = -\frac{\delta(\pi)}{2}\log\log x + O(\log\log\log|d|).$$

□

## 7.2 The Frequency of Large Values of $\prod_{1\le i\le k}|L(\frac{1}{2},\pi_i\otimes\chi_d)|$

In this section, we state and prove a value distribution result for a linear combination of distinct primitive $L$-functions twisted by quadratic Dirichlet characters which will be used to prove Theorem 1.6.1.

Let $\chi_d$ be a primitive quadratic character of conductor $|d|$, and let $L(s,\pi_1)$,. . ., $L(s,\pi_r)$ be $r$ distinct primitive $L$-functions (as in Theorem 1.6.1) of degrees $m_1,\dots,m_r$, respectively, and let

$$B = k_1 m_1 + \cdots + k_r m_r + 1.$$

Let $\mathcal{N}(X,T)$ be the set defined in (7.0.3), and let

$$W = (k_1^2 + \cdots + k_r^2) \log\log X.$$

**Proposition 7.2.1.** *Let $d$ denote a fundamental discriminant, and let $\chi_d$ be a primitive quadratic Dirichlet character of conductor $|d|$. Let $L(s,\pi_1), \ldots, L(s,\pi_r)$ be L-functions attached to distinct self-contragredient irreducible cuspidal automorphic representations, $\pi_j$, of $\mathrm{GL}(m_j)$ over $\mathbb{Q}$ each with unitary central character, and assume that the twisted L-functions $L(s,\pi_1\otimes\chi_d), \ldots, L(s,\pi_r\otimes\chi_d)$ satisfy the Generalized Riemann Hypothesis and the Ramanujan-Petersson Conjecture. If $\sqrt{W} \le V \le \frac{W}{B^2}$, we have*

$$\mathrm{meas}(\mathcal{N}(X,V)) \ll X \exp\Big(-\frac{V^2}{2W}\Big(1-\frac{4}{\log W}\Big)\Big);$$

*if $\frac{W}{B^2} < V < \frac{1}{2B^2} W \log W$, we have*

$$\mathrm{meas}(\mathcal{N}(X,V)) \ll X \exp\Big(-\frac{V^2}{2W}\Big(1-\frac{7B^2 V}{4W\log W}\Big)^2\Big);$$

*and if $\frac{1}{2B^2} W \log W < V$, we have*

$$\mathrm{meas}(\mathcal{N}(X,V)) \ll X \exp\Big(-\frac{1}{256B^2} V \log V\Big).$$

*for any $k_1, \ldots, k_r > 0$ when $X$ is sufficiently large.*

*Proof.* The proof is similar to the proof of Proposition 4.3.1. Let $\lambda = \lambda_0 < 1/2$ and $\varepsilon < (1-2\lambda_0)/3$. Choosing $x = (\log X)^{1-\varepsilon}$, it follows from Lemma 7.1.2 and (4.1.1)

$$\log|L(\tfrac{1}{2}, \pi\otimes\chi_d)| \le \frac{3m}{4}\frac{\log X}{\log\log X}$$

for sufficiently large $X$. Therefore, we see that

$$k_1 \log|L(\tfrac{1}{2}, \pi_1 \otimes \chi_d)| + \cdots + k_r \log|L(\tfrac{1}{2}, \pi_r \otimes \chi_d)| \le \frac{3(k_1m_1 + \cdots + k_rm_r)}{4} \frac{\log X}{\log\log X}$$

when $X$ is large. Hence, we may assume that

$$\sqrt{W} \le V \le \frac{3(B-1)}{4} \frac{\log X}{\log\log X}$$

while proving the proposition. As in the proof of Proposition 4.3.1, define a parameter $A$ as

$$A = \begin{cases} \dfrac{B}{2} \log W, & \text{if } \sqrt{W} \le V \le \frac{W}{B^2}, \\[2ex] \dfrac{1}{2BV} W \log W, & \text{if } \frac{W}{B^2} \le V \le \frac{1}{2B^2} W \log W, \\[2ex] B, & \text{if } V > \frac{1}{2B^2} W \log W, \end{cases}$$

and let $x = X^{A/V}$ and $z = x^{1/\log\log X}$. Choosing $\lambda = 1/2$ in (7.1.2), we deduce that

$$\begin{aligned} k_1 \log\left|L(\tfrac{1}{2}, \pi_1 \otimes \chi_d)\right| &+ \cdots + k_r \log\left|L(\tfrac{1}{2}, \pi_r \otimes \chi_d)\right| \\ &\le |S_1(d)| + |S_1^\star(d)| + \Re\{S_2(d)\} + \frac{3(B-1)}{4} \frac{V}{\log A} + O(1), \end{aligned}$$

where

$$S_1(d) = \sum_{p \le z} \frac{(k_1\Lambda_{\pi_1}(p) + \cdots + k_r\Lambda_{\pi_r}(p))\, \chi_d(p)}{p^{\frac{1}{2} + \frac{\lambda}{\log x}} \log p} \frac{\log(x/p)}{\log x},$$

$$S_1^\star(d) = \sum_{z < p \le x} \frac{(k_1\Lambda_{\pi_1}(p) + \cdots + k_r\Lambda_{\pi_r}(p))\, \chi_d(p)}{p^{\frac{1}{2} + \frac{\lambda}{\log x}} \log p} \frac{\log(x/p)}{\log x},$$

and

$$S_2(d) = \sum_{p^2 \le x} \frac{(k_1\Lambda_{\pi_1}(p^2) + \cdots + k_r\Lambda_{\pi_r}(p^2))\, \chi_d(p^2)}{p^{1 + \frac{2\lambda}{\log x}} \log p^2} \frac{\log(x/p^2)}{\log x}.$$

The contribution from the prime powers $p^j$ for which $j > 2$ is $O(1)$. Indeed, the Ramanujan-Petersson Conjecture implies that

$$|k_1\Lambda_{\pi_1}(p^j) + \cdots + k_r\Lambda_{\pi_r}(p^j)| \ll (k_1m_1 + \cdots + k_rm_r)(\log p).$$

Hence

$$\begin{aligned}\sum_{\substack{p^j \le x\\ j>2}} \frac{|k_1\Lambda_{\pi_1}(p^j) + \cdots + k_r\Lambda_{\pi_r}(p^j)|\,|\chi_d(p^j)|}{p^{j(\frac{1}{2}+\frac{\lambda}{\log x}+it)}\log p^j}\frac{\log(x/p^j)}{\log x} &\ll \sum_{\substack{p^j \le x\\ j>2}} \frac{|k_1\Lambda_{\pi_1}(p^j) + \cdots + k_r\Lambda_{\pi_r}(p^j)|}{jp^{j/2}\log p}\\ &\ll (B-1)\sum_{\substack{p^j \le x\\ j>2}} \frac{1}{p^{j/2}}\\ &\ll 1,\end{aligned}$$

since $|\chi_d(p^j)| \le 1$ for all $j$. Noting that $|d| \le X$, Lemma 7.1.3 together with partial summation provides that

$$S_2(d) = -\left(\frac{k_1\delta(\pi_1) + \cdots + k_r\delta(\pi_r)}{2}\right)\log\log X + O(\log\log\log X).$$

Thus, we have

$$\begin{aligned}&k_1\log|L(\tfrac{1}{2},\pi_1\otimes\chi_d)| + \cdots + k_r\log|L(\tfrac{1}{2},\pi_r\otimes\chi_d)|\\ &\qquad\le |S_1(d)| + |S_1^\star(d)| - \left(\frac{k_1\delta(\pi_1) + \cdots + k_r\delta(\pi_r)}{2}\right)\log\log X\\ &\qquad\qquad + \frac{3(B-1)}{4}\frac{V}{A} + O(\log\log\log X).\end{aligned}$$

Define

$$V_1 := V\left(1 - \frac{7(B-1)}{8A}\right) \quad\text{and}\quad V_1^\star := \frac{(B-1)V}{8A},$$

and note that if $d \in \mathcal{N}(X,V)$, then at least one of the following inequalities holds:

$$|S_1(d)| \geq V_1 \quad \text{or} \quad |S_1^\star(d)| \geq V_1^\star.$$

If we define

$$N_1(X,V_1) := \text{meas}\{|d| \leq X : |S_j(d)| \geq V_1\}$$

and $N_1^\star(X,V_1^\star)$ similarly, we can bound $N_1(X,V_1)$ and $N_1^\star(X,V_1^\star)$ using Lemma 7.1.1 since Chebyshev's inequality implies that

$$N_1(X,V_j) \leq (V_1)^{-2\ell} \sum_{|d|\leq X}^{\flat} |S_j(d)|^{2\ell}$$

and

$$N_1^\star(X,V_1^\star) \leq (V_1^\star)^{-2\ell} \sum_{|d|\leq X}^{\flat} |S_1^\star(d)|^{2\ell}$$

for any non-negative integer $\ell$.

Let us first estimate $N_1(X,V_1)$. Letting $\ell$ be any natural number such that $\ell \leq \frac{\log(X^{1/2}\log X)}{\log z} = \frac{V\log\log X}{2A}(1-\frac{2\log\log X}{\log X})$, Lemma 7.1.1 and (4.3.3) imply that

$$\begin{aligned}\sum_{|d|\leq X}^{\flat} |S_1(t)|^{2\ell} &\ll X\frac{(2\ell)!}{\ell!2^\ell}\left(\sum_{p\leq z}\frac{|k_1\Lambda_{\pi_1}(p)+\cdots+k_r\Lambda_{\pi_r}(p)|^2}{p\log^2 p}\right)^\ell \\ &\ll X\left(\frac{2\ell W}{e}\right)^\ell.\end{aligned}$$

Thus we have

$$N_1(X,V_1) \ll X\left(\frac{2\ell W}{eV_1^2}\right)^\ell.$$

We consider separately the two cases where $V \leq \frac{W^2}{B^4}$ and $V > \frac{W^2}{B^4}$. In the first case, we choose $\ell = \lfloor \frac{V_1^2}{2W} \rfloor$ and find that

$$N_1(X,V_1) \ll X\exp\Big(-\frac{V_1^2}{2W}\Big).$$

In the case where $V > \frac{W^2}{B^4}$, we choose $\ell = \lfloor 10V \rfloor$ and find that

$$N_1(X, V_1) \ll X \exp(-5V \log V).$$

Hence

$$N_1(X, V_1) \ll X \exp\Big(-\frac{V_1^2}{2W}\Big) + X \exp(-5V \log V) \tag{7.2.1}$$

for all $V$.

Next, we find an upper bound for $N_1^\star(X, V_1^\star)$. For any natural number $\ell \le \frac{V}{2A}(1 - \frac{\log\log X}{\log X})$, Lemma 7.1.1 and (4.3.3) imply that

$$\sum_{|d| \le X}^{\flat} |S_1^\star(d)|^{2\ell} \ll X \left(2\ell(k_1^2 + \cdots + k_r^2) \log\log\log X\right)^{\ell}$$

when $X$ is large. Choosing $\ell = \lfloor \frac{V}{2A} - 1 \rfloor$, we have that

$$\begin{aligned} N_1^\star(X, V_1^\star) &\ll X \left(\frac{16A}{(B-1)V}\right)^{2\ell} \left\{2\ell(k_1^2 + \cdots + k_r^2) \log\log\log X\right\}^{\ell} \\ &\ll X V^{-2\ell} A^{2\ell} (2\ell \log\log\log X)^{\ell} \\ &\ll X V^{-V/2A+1} \left[V \left(A \log\log\log X\right)^{V/2A-1}\right] \\ &\ll X V^{-V/4A}. \end{aligned}$$

Here we have used the fact that

$$V \left(A \log\log\log X\right)^{V/2A-1} \ll V^{V/4A}.$$

Thus,

$$N_1^\star(X, V_1^\star) \ll X \exp\left(-\frac{V}{4A} \log V\right). \tag{7.2.2}$$

The proposition now follows by combining (7.2.1) and (7.2.2).

□

## 7.3 The Proof of Theorem 1.6.1

We now use Proposition 7.2.1 and (7.0.2) to prove Theorem 1.6.1.

*Proof.* Proposition 7.2.1 implies that

$$\mathcal{N}(X,V) \ll \begin{cases} X\,(\log X)^{\varepsilon} \exp\left(-\frac{V^2}{2W}\right), & \text{if } 3 \leq V \leq \frac{512W}{B^2}, \\ X\,(\log X)^{\varepsilon} \exp\left(-\frac{4V}{B^2}\right), & \text{if } V > \frac{512W}{B^2}. \end{cases}$$

Theorem 1.6.1 now follows by inserting these bounds into (7.0.2). □

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