Large gaps between zeros of Dedekind zeta-functions of quadratic number fields

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Definition of the Dedekind Zeta-function of K

Let K be a number field, and let $\mathcal{O}_K$ be its ring of integers. Then

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}, \quad \Re(s) > 1,$$

where I and $\mathfrak{p}$ run over the nonzero ideals and prime ideals of $\mathcal{O}_K$, respectively.

Let $K = \mathbb{Q}[\sqrt{d}]$, where d is a fundamental discriminant. Then

$$\zeta_K(s) = \zeta(s) L(s, \chi_d),$$

where $\zeta(s)$ is the Riemann zeta-function and $L(s, \chi_d)$ is the Dirichlet L-function associated to χ_d, the Kronecker symbol of the discriminant d.

Zeros of $\zeta_K(s)$

Let $\rho = \beta + i\gamma$ denote a nontrivial zero of $\zeta_K(s)$. Then

$$N_K(T) := \sum_{0 < \gamma < T} 1 \sim \frac{T}{\pi} \log \sqrt{|d|} T,$$

as $T \to \infty$. Define the sequence

$$0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_n \leq \ldots$$

of non-negative ordinates of zeros of $\zeta_K(s)$ in increasing order. Then the average size of $\gamma_{n+1} - \gamma_n$ is $\dfrac{\pi}{\log \sqrt{|d|}\gamma_n}$.

It is conjectured that

$$\liminf_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{(\pi / \log \sqrt{|d|}\gamma_n)} = 0$$

and

$$\limsup_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{(\pi / \log \sqrt{|d|}\gamma_n)} = \infty.$$

Note: One can work out an analogue of Montgomery's pair correlation conjecture for the zeros of $\zeta_K(s)$.

Theorem 1: Large Gaps Result (T.-B., 2012)

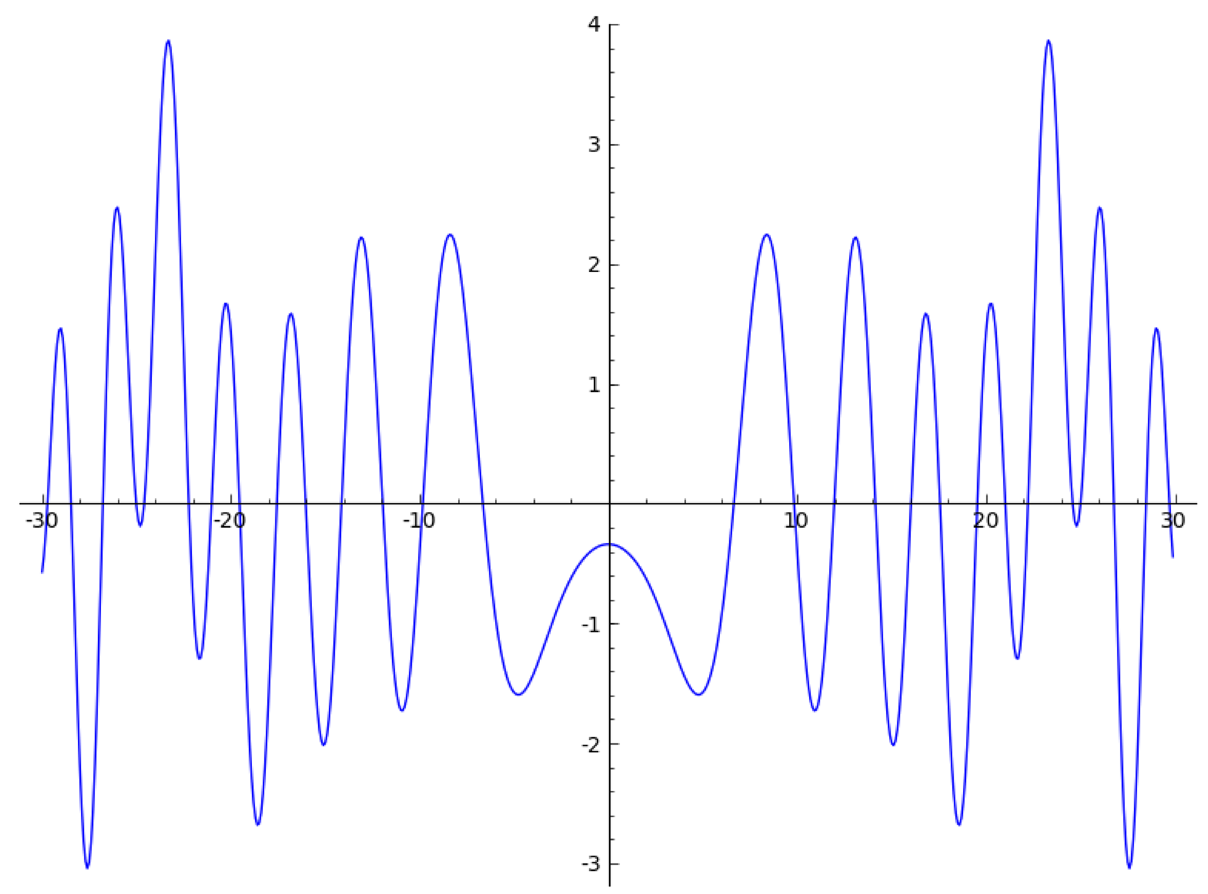
Let $\{\tilde{\gamma}_n\}$ be the subsequence of ordinates of zeros of $\zeta_K(s)$ on the critical line. Then

$$\limsup_{n \to \infty} \frac{\tilde{\gamma}_{n+1} - \tilde{\gamma}_n}{(\pi / \log \sqrt{|d|}\tilde{\gamma}_n)} \geq \sqrt{6} = 2.4494\ldots.$$

Assuming the generalized Riemann hypothesis for $\zeta_K(s)$, we have

$$\limsup_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{(\pi / \log \sqrt{|d|}\gamma_n)} \geq \sqrt{6} = 2.4494\ldots.$$

Visualizing the Zeros of $\zeta_K(\frac{1}{2} + it)$



Graph of the Z-function along the critical line, where $K = \mathbb{Q}[\sqrt{5}]$. (www.lmfdb.org)

Hall's Method: Wirtinger's Inequality

Let $f : [a, b] \to \mathbb{C}$ be continuously differentiable, and suppose that $f(a) = f(b) = 0$. Then

$$\int_a^b |f(t)|^2 dt \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(t)|^2 dt.$$

Theorem 2: Mixed Moments (T.-B., 2012)

Let μ, ν be non-negative integers. Then

$$\int_T^{2T} \zeta_K^{(\mu)}(\tfrac{1}{2} + it) \overline{\zeta_K^{(\nu)}(\tfrac{1}{2} + it)} dt \sim C_d(\mu, \nu) T (\log T)^{\mu + \nu + 2},$$

as $T \to \infty$. Here the constant $C_d(\mu, \nu)$ is explicitly given.

Note: The cases $\mu = \nu = 0$ and $\mu = \nu = 1$ were proved in 5. and 6., respectively.

Theorem 1 - Proof Sketch

We choose a test function that is nonvanishing and of modulus 1 and define

$$f(t) := \begin{pmatrix} \text{test} \\ \text{function} \end{pmatrix} \cdot \zeta_K(\tfrac{1}{2} + it).$$

Let γ^+ and γ denote consecutive ordinates of zeros of $\zeta_K(s)$ on the critical line. Applying Wirtinger's Inequality and summing over the zeros between T and $2T$, we deduce that

$$\int_T^{2T} |f(t)|^2 dt \leq \max_{T \leq \gamma \leq 2T} \frac{(\gamma^+ - \gamma)^2}{\pi^2} \int_T^{2T} |f'(t)|^2 dt + \begin{pmatrix} \text{small} \\ \text{error} \end{pmatrix}.$$

For a certain family of test functions, Theorem 2 allows us to asymptotically evaluate the above integrals. Optimizing within this family leads to Theorem 1. □

Ingham's Shifted Moment (1926)

For $s = \frac{1}{2} + it, \tau = t/2\pi$, and $|\alpha|, |\beta| \leq \frac{1}{2}$, we have

$$\int_0^T \zeta(s+\alpha)\zeta(1-s-\beta) dt$$
$$\sim \int_0^T \left\{\zeta(1+\alpha-\beta) + \tau^{\beta-\alpha}\zeta(1+\beta-\alpha)\right\} dt.$$

This 'shifted' moment allows one to deduce lower order terms and moments of derivatives (via differentiation and Cauchy's integral formula).

Shifted Moment Conjecture for $\zeta_K(s)$

For $s = \frac{1}{2} + it, \tau = t/2\pi$, and $|\alpha|, |\beta| \leq \frac{1}{2}$, we have

$$\int_0^T \zeta_K(s+\alpha)\zeta_K(1-s-\beta) dt$$
$$\sim \int_0^T \Big\{C_1 \zeta_K^2(1+\alpha-\beta) + C_2 \tau^{2\beta - 2\alpha} \zeta_K^2(1+\beta-\alpha)$$
$$+ C_3 \tau^{\beta-\alpha} L^2(1, \chi_d) \zeta(1+\alpha-\beta)\zeta(1+\beta-\alpha)\Big\} dt + I,$$

where

$$I = \int_0^T C_4 \tau^{\beta-\alpha} L^2(1, \chi_d) L(1+\alpha-\beta, \chi_d) L(1+\beta-\alpha, \chi_d) \, dt.$$

The constants C_1, C_2, C_3 and C_4 depend on d, α, β and can be written in terms of absolutely convergent Euler products. This conjecture is obtained by modifying heuristics in 1. and 3..

Theorem 3: Shifted Moment (T.-B., 2012)

For $|\alpha|, |\beta| \leq 1/\log T$ and $|d| \leq \log T$, the above conjecture is true with an error of $O(|d|^\varepsilon T \log^{3/2} T)$.

Note: Differentiating with respect to α and β and then letting $\alpha, \beta \to 0$, we can deduce Theorem 2 from Theorem 3.

References

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