

# Large gaps between zeros of Dedekind zeta-functions of quadratic number fields

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## Definition of the Dedekind Zeta-function of $K$

Let  $K$  be a number field, and let  $\mathcal{O}_K$  be its ring of integers. Then

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{N(I)^s} = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}, \quad \Re(s) > 1,$$

where  $I$  and  $\mathfrak{p}$  run over the nonzero ideals and prime ideals of  $\mathcal{O}_K$ , respectively.

Let  $K = \mathbb{Q}[\sqrt{d}]$ , where  $d$  is a fundamental discriminant. Then

$$\zeta_K(s) = \zeta(s)L(s, \chi_d),$$

where  $\zeta(s)$  is the Riemann zeta-function and  $L(s, \chi_d)$  is the Dirichlet  $L$ -function associated to  $\chi_d$ , the Kronecker symbol of the discriminant  $d$ .

## Zeros of $\zeta_K(s)$

Let  $\rho = \beta + i\gamma$  denote a nontrivial zero of  $\zeta_K(s)$ . Then

$$N_K(T) := \sum_{0 < \gamma < T} 1 \sim \frac{T}{\pi} \log \sqrt{|d|}T,$$

as  $T \rightarrow \infty$ . Define the sequence

$$0 \leq \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n \leq \dots$$

of non-negative ordinates of zeros of  $\zeta_K(s)$  in increasing order. Then the average size of  $\gamma_{n+1} - \gamma_n$  is  $\frac{\pi}{\log \sqrt{|d|}\gamma_n}$ .

It is conjectured that

$$\liminf_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{(\pi / \log \sqrt{|d|}\gamma_n)} = 0$$

and

$$\limsup_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{(\pi / \log \sqrt{|d|}\gamma_n)} = \infty.$$

**Note:** One can work out an analogue of Montgomery's pair correlation conjecture for the zeros of  $\zeta_K(s)$ .

## Theorem 1: Large Gaps Result (T.-B., 2012)

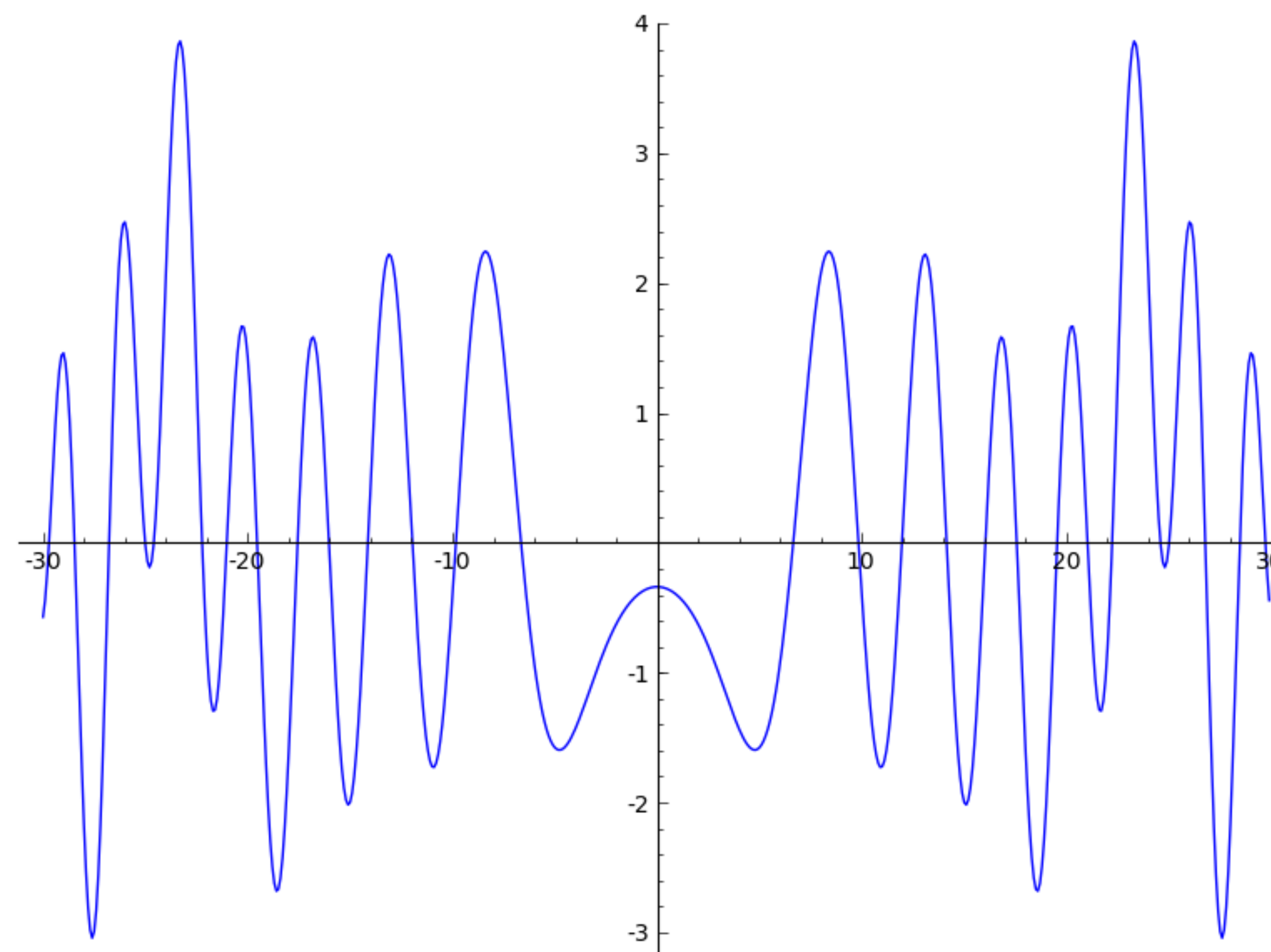
Let  $\{\tilde{\gamma}_n\}$  be the subsequence of ordinates of zeros of  $\zeta_K(s)$  on the critical line. Then

$$\limsup_{n \rightarrow \infty} \frac{\tilde{\gamma}_{n+1} - \tilde{\gamma}_n}{(\pi / \log \sqrt{|d|}\tilde{\gamma}_n)} \geq \sqrt{6} = 2.4494\dots$$

Assuming the generalized Riemann hypothesis for  $\zeta_K(s)$ , we have

$$\limsup_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{(\pi / \log \sqrt{|d|}\gamma_n)} \geq \sqrt{6} = 2.4494\dots$$

## Visualizing the Zeros of $\zeta_K(\frac{1}{2} + it)$



Graph of the  $Z$ -function along the critical line, where  $K = \mathbb{Q}[\sqrt{5}]$ .

(www.lmfdb.org)

## Hall's Method: Wirtinger's Inequality

Let  $f : [a, b] \rightarrow \mathbb{C}$  be continuously differentiable, and suppose that  $f(a) = f(b) = 0$ . Then

$$\int_a^b |f(t)|^2 dt \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(t)|^2 dt.$$

## Theorem 2: Mixed Moments (T.-B., 2012)

Let  $\mu, \nu$  be non-negative integers. Then

$$\int_T^{2T} \zeta_K^{(\mu)}\left(\frac{1}{2} + it\right) \overline{\zeta_K^{(\nu)}\left(\frac{1}{2} + it\right)} dt \sim C_d(\mu, \nu) T (\log T)^{\mu+\nu+2},$$

as  $T \rightarrow \infty$ . Here the constant  $C_d(\mu, \nu)$  is explicitly given.

**Note:** The cases  $\mu = \nu = 0$  and  $\mu = \nu = 1$  were proved in 5. and 6., respectively.

## Theorem 1 - Proof Sketch

We choose a test function that is nonvanishing and of modulus 1 and define

$$f(t) := \left(\begin{matrix} \text{test} \\ \text{function} \end{matrix}\right) \cdot \zeta_K\left(\frac{1}{2} + it\right).$$

Let  $\gamma^+$  and  $\gamma$  denote consecutive ordinates of zeros of  $\zeta_K(s)$  on the critical line. Applying Wirtinger's Inequality and summing over the zeros between  $T$  and  $2T$ , we deduce that

$$\int_T^{2T} |f(t)|^2 dt \leq \max_{T \leq \gamma \leq 2T} \frac{(\gamma^+ - \gamma)^2}{\pi^2} \int_T^{2T} |f'(t)|^2 dt + \left(\begin{matrix} \text{small} \\ \text{error} \end{matrix}\right).$$

For a certain family of test functions, Theorem 2 allows us to asymptotically evaluate the above integrals. Optimizing within this family leads to Theorem 1.  $\square$

## Ingham's Shifted Moment (1926)

For  $s = \frac{1}{2} + it$ ,  $\tau = t/2\pi$ , and  $|\alpha|, |\beta| \leq \frac{1}{2}$ , we have

$$\int_0^T \zeta(s+\alpha)\zeta(1-s-\beta) dt \\ \sim \int_0^T \left\{ \zeta(1+\alpha-\beta) + \tau^{\beta-\alpha} \zeta(1+\beta-\alpha) \right\} dt.$$

This 'shifted' moment allows one to deduce lower order terms and moments of derivatives (via differentiation and Cauchy's integral formula).

## Shifted Moment Conjecture for $\zeta_K(s)$

For  $s = \frac{1}{2} + it$ ,  $\tau = t/2\pi$ , and  $|\alpha|, |\beta| \leq \frac{1}{2}$ , we have

$$\int_0^T \zeta_K(s+\alpha)\zeta_K(1-s-\beta) dt \\ \sim \int_0^T \left\{ C_1 \zeta_K^2(1+\alpha-\beta) + C_2 \tau^{2\beta-2\alpha} \zeta_K^2(1+\beta-\alpha) \right. \\ \left. + C_3 \tau^{\beta-\alpha} L^2(1, \chi_d) \zeta(1+\alpha-\beta) \zeta(1+\beta-\alpha) \right\} dt + I,$$

where

$$I = \int_0^T C_4 \tau^{\beta-\alpha} L^2(1, \chi_d) L(1+\alpha-\beta, \chi_d) L(1+\beta-\alpha, \chi_d) dt.$$

The constants  $C_1, C_2, C_3$  and  $C_4$  depend on  $d, \alpha, \beta$  and can be written in terms of absolutely convergent Euler products. This conjecture is obtained by modifying heuristics in 1. and 3..

## Theorem 3: Shifted Moment (T.-B., 2012)

For  $|\alpha|, |\beta| \leq 1/\log T$  and  $|d| \leq \log T$ , the above conjecture is true with an error of  $O(|d|^\epsilon T \log^{3/2} T)$ .

**Note:** Differentiating with respect to  $\alpha$  and  $\beta$  and then letting  $\alpha, \beta \rightarrow 0$ , we can deduce Theorem 2 from Theorem 3.

## References

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